# Spectral Theorem For Unbounded Operators

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# Contents

# Einleitung

Ein Spektraltheorem oder Spektralkalkül gibt der Anschauung, dass man Operatoren auf einem Hilbertraum in Funktionen einsetzen kann, eine rigorose mathematische Grundlage. Dass man Operatoren in Polynome einsetzen kann, und immer noch sinnvolle Ausdrücke entstehen, liegt auf der Hand. Wie verhält es sich jedoch mit stetigen oder gar messbaren Funktionen? Ergibt der Ausdruck $f(T)$ für beliebige Funktionen $f$ und Operatoren $T$ überhaupt Sinn? Was ist $\mathfrak{D}(f(T))$ und $\mathfrak{R}(f(T))$? Ist $f(T)$ dicht definiert? Diese Fragen möchte ich in dieser Arbeit, soweit es geht, beantworten.

Das Ziel dieser Arbeit ist ein Spektraltheorem für unbeschränkte normale Operatoren. Aus diversen Vorlesungen an der Universität Bonn waren mir Spektraltheoreme für explizite Klassen von Operatoren bekannt, zum Beispiel für kompakte, selbstadjungerte Operatoren. Als ich in einem Seminar eine Variante für unbeschränkte Operatoren benutzen musste, entschied ich mich mehr mit diesem Thema zu beschäftigen. Diese Arbeit ist an Studenten der Mathematik oder Physik gerichtet, welche eine mathematisch rigorose Formulierung des Spektraltheorems für unbeschränkte Operatoren kennen lernen möchten.

In meiner Bachelorarbeit wird in Kapitel 2 mit Hilfe des Gelfandschen Transformationsatzes ein Spektraltheorem für beschränkte normale Operatoren bewiesen. In Kapitel 3 wird versucht, die Methoden des voran gegangenen Kapitels auf unbeschränkte Operatoren zu erweitern. Um dies zu tun, muss das Spektraltheorem auf messbare Funktionen erweitert werden. Dazu wird in Kapitel 4 beschrieben wie sich Erweiterungen des Spektraltheorems auf messbare Funktionen verhalten, das heißt, welche Klassen von Operatoren erhalten werden. Abschließend werden die Ergebnisse der vorherigen Kapitel benutzt, um das Spektraltheorem für unbeschränkte normale Operatoren zu beweisen, und in Kapitel 6 mit ein paar Beispielen erläutert.

In der Literatur werden zum Beweis von Spektraltheoremen oft, operatorwertige Maße auf bestimmten $\sigma$–Algebren genutzt. Dies wird in dieser Arbeit explizit nicht genutzt. Welche der Möglichkeiten man benutzt, bleibt der eigenen Vorliebe überlassen. Der Riesz–Markov–Kakutani Darstellungssatz ([4, Theorem 6.3.4]) zeigt, dass beide Herangehensweisen lediglich zwei Seiten der gleichen Medaille sind.

Ich möchte mich an dieser Stelle bei meinem Betreuer Herrn Professor Lesch bedanken. Erst durch seine Betreuung und stetige Erreichbarkeit wurde diese Arbeit ermöglicht. Weiterhin danke ich meinen Kommilitonen für das Korrekturlesen dieser Arbeit. Zum Schluss möchte ich meinen Eltern für die Unterstützung meines Studiums danken. Ohne Sie, wäre diese nie zu Stande gekommen.

# 1 Preliminaries

In this section, I want to state a few theorems and definitions, which will be used later on. If no proof is given, a reference will be stated nevertheless. For an unital algebra $\mathfrak{A}$, we denote the invertible elements by $\mathbf{GL}(\mathfrak{A})$.

**Definition 1.1** (Spectrum)**.** Let $\mathfrak{A}$ be a unital commutative Banach algebra. For $A$ an element of $\mathfrak{A}$, the *spectrum* of $A$ in $\mathfrak{A}$, is defined as

$$\mathrm{Sp}_{\mathfrak{A}}(A) := \{z \in \mathbb{C} \mid A - zI \notin \mathbf{GL}(\mathfrak{A})\}.$$

The *spectrum* of $\mathfrak{A}$ is defined as

$$\mathrm{Sp}(\mathfrak{A}) := \{\chi : \mathfrak{A} \to \mathbb{C} \mid \chi \in \mathrm{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}(\mathfrak{A}, \mathbb{C}), \chi \neq 0\}.$$

**Proposition 1.2.** *Using the notation above,*

$$\mathrm{Sp}_{\mathfrak{A}}(A) = \left\{\chi(A) \;\middle|\; \chi \in \mathrm{Sp}\,\mathfrak{A}\right\}.$$

*In other words, the spectrum of an element is the image of that element under the spectrum of the algebra.*

The proof can be found in [4, Ch. 4.2]. By $\mathfrak{A}'$ we denote the dual space $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{A}, \mathbb{C})$, endowed with the weak-$*$ topology.

**Theorem 1.3** (Gelfand–Naimark)**.** *Let $\mathfrak{A}$ be a commutative $C^*$-algebra. If* $\mathrm{Sp}(\mathfrak{A})$ *is equipped with the supspace topology of $\mathfrak{A}'$, it becomes a compact space, with a canonical isometric, involutive, surjective algebra homomorphism*

$$\mathcal{G} : \mathfrak{A} \to \mathrm{C}(\mathrm{Sp}(\mathfrak{A})),\ A \mapsto (\hat{A} := \mathcal{G}(A) : \mathrm{Sp}(\mathfrak{A}) \to \mathbb{C},\ \gamma \mapsto \gamma(A)).$$

The proof can be found in [4, Ch. 4.3]. The map in the theorem is the so called Gelfand transform.

The reader not familiar with closed operators is advised to revisit the basic definitions found in [1, Ch. 10]. When we speak of a linear operator, we always mean a linear operator. Let $T$ be a operator on some Hilbert space $\mathcal{H}$. We will adopt the notation from [1, Ch. 10], that is, $T$ is not even assumed to be defined for any non zero element. Later on, we will only concern ourselfs with densly defined, closed operators. By $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ we denote the domain and the range of the operator, respectively. The set of bounded linear operators will be called $\mathcal{B}(\mathcal{H})$, in contrast to $\mathscr{L}(\mathcal{H})$, which is the set of all linear operators. The set of closed operators is denoted by $\mathcal{C}(\mathcal{H})$.

**Definition 1.4.** If $T$ and $S$ are operators on $\mathcal{H}$, we say $T$ *extends* $S$, if $\mathfrak{D}(S) \subset \mathfrak{D}(T)$ and $Tx = Sx$ for all $x$ in $\mathfrak{D}(S)$. We write $S \subset T$.

Let $S$ and $T$ be operators on $\mathcal{H}$. By $S + T$, we denote the operator with domain $\mathfrak{D}(S) \cap \mathfrak{D}(T)$ and rule $(S + T)x = Sx + Tx$. Note that this does not give $\mathscr{L}(\mathcal{H})$ the structure of a vector space, for $(S + T) - T \neq S$ since $\mathfrak{D}(T) \cap \mathfrak{D}(S) \neq \mathfrak{D}(S)$. $TS$ is defined to be the operator with domain $S^{-1}\mathfrak{D}(T)$. The reader should be aware, that with the just defined operations, $\mathscr{L}(\mathcal{H})$ does not admit the structure of an algebra, and as previously remarked, not even that of a vector space. This is one of the reasons, why one has to be careful while working with unbounded operators.

Since closed operators are not defined on all of $\mathcal{H}$, there is no obvious definition of an inverse to such an operator. However, we have the following

**Definition 1.5.** Let $T$ denote a closed operator on $\mathcal{H}$. We say that $T$ is *boundedly invertible* if $T : \mathfrak{D}(T) \to \mathfrak{R}(T) = \mathcal{H}$ is a bijection, and $T^{-1} : \mathcal{H} \to \mathfrak{D}(T)$ is continuous. $T^{-1}$ is called the *bounded inverse* of $T$. As $T^{-1}$ is continuous, the closed graph theorem implies the boundedness of $T^{-1}$.

**Remark 1.6.** If $T$ is boundedly invertible, then the inverse is unique.

**Lemma 1.7.** *If $T \in \mathcal{C}(\mathcal{H})$, $S \in \mathcal{B}(\mathcal{H})$ and $TS = \mathrm{id}$ then $S$ is the bounded inverse of $T$.*

*Proof.* It remains to show that $T$ is a bijection. Surjectivity is obvious. Note that $\ker(S) = 0$, which implies that $\ker(T) = 0$ as well. □

For $A, B, \ldots \in \mathfrak{A}$, we denote by $\langle A, B, \ldots \rangle$ the $C^*$-subalgebra of $\mathfrak{A}$, generated by the elements $A, B, \ldots$

# 2 Spectral Theorem For Bounded Opearators

In this chapter, we use the Gelfand transform to prove a continuous spectral theorem for bounded normal operators, and some auxiliary results about the spectrum of $C^*$-algebras.

**Proposition 2.1.** *Let $\mathfrak{A}$ be a $C^*$-subalgebra of the $C^*$-algebra $\mathfrak{B}$. Let $A$ be an element of $\mathfrak{A}$, which is invertible in $\mathfrak{B}$. Then $A$ is already invertible in $\mathfrak{A}$. In other words*

$$\mathrm{Sp}_{\mathfrak{A}}(A) = \mathrm{Sp}_{\mathfrak{B}}(A) \text{ for all } A \text{ in } \mathfrak{A}.$$

*Proof.* First assume $A = A^*$. We have $\mathrm{Sp}_{\mathfrak{A}}(A) \subset \mathbb{R}$, implying that $(A + i\lambda I)$ is invertible in $\mathfrak{A}$, for all $A$ in $\mathfrak{A}$. As

$$\lim_{\lambda \to 0} (A + i\lambda I) = A,$$

by continuity of the inverse map, and the assumption that $A$ is invertible, we get

$$\lim_{\lambda \to 0} (A + i\lambda I)^{-1} = A^{-1}.$$

Because $(A + i\lambda I)^{-1}$ is an element of $\mathfrak{A}$ for all $\lambda \neq 0$, the statement holds for self adjoint $A$, as $\mathfrak{A}$ is a closed subalgebra.

For more general $A$, we consider the self adjoint element $A^*A$, with inverse $(A^*A)^{-1} = A^{-1}(A^{-1})^*$. Since $\mathfrak{A}$ is an involutive algebra, $A^*A$ is an element of $\mathfrak{A}$, which implies that $A$ is left-invertible in $\mathfrak{A}$, with inverse $(A^*A)^{-1}A^*$. Using the same argument with the normal element $AA^*$, one gets the right-invertibility of $A$. Thus $A$ is invertible, and the inverses coincide. □

**Corollary 2.2.** *Let $\mathfrak{A}$ be a $C^*$-subalgebra of $\mathscr{L}(\mathcal{H})$, $T \in \mathfrak{A}$. Then*

$$\mathrm{Sp}_{\mathfrak{A}}(T) = \mathrm{Sp}_{\mathscr{L}(\mathcal{H})}(T) = \mathrm{Sp}(T).$$

**Proposition 2.3** (Functional calculus for normal elements)**.** *Let $\mathfrak{B}$ be a $C^*$-algebra with unit, and $A$ a normal element. Then the algebra $\mathfrak{A} = \langle A, I \rangle$ generated by $A$ and the identity $I$, is a normal involutive subalgebra, which is isomorphic to* $\mathrm{C}(\mathrm{Sp}\, A)$*, where* $\mathrm{Sp}(A)$ *is identified with* $\mathrm{Sp}(\mathfrak{A})$ *via the Gelfandtransform.*

*Proof.* First, we show that $\mathcal{G}_A : \mathrm{Sp}(\mathfrak{A}) \to \mathbb{C}$ is injective. Let $\chi_1, \chi_2$ be elements of $\mathrm{Sp}(\mathfrak{A})$. If $\mathcal{G}_A(\chi_1) = \mathcal{G}_A(\chi_2) = \chi_2(A) = \chi_1(A)$, then also $\chi_1(A^*) = \chi_2(A^*)$. Since $\chi_1(I) = \chi_2(I) = 1$, we see that $\chi_1 = \chi_2$ on all polynomials in $A$ and $A^*$. Because $\chi_1$, $\chi_2$ are continous, they have to coincide on $\mathfrak{A}$.

By Proposition 1.2, $\mathcal{G}_A$ is surjective. Hence, $\mathcal{G}_A$ is a continous bijection from $\mathrm{Sp}(\mathfrak{A})$ to $\mathrm{Sp}(A)$. As $\mathrm{Sp}(\mathfrak{A})$ is compact, $\mathcal{G}_A$ is a homeomorphism. By the theorem of Gelfand–Neimark

$$\mathcal{G} : \mathfrak{A} \to \mathrm{C}(\mathrm{Sp}\, \mathfrak{A})$$

is an isomorphism. We get the following commutative diagram, which yields the result

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{B \mapsto \mathcal{G}_B} & \mathrm{C}(\mathrm{Sp}\, \mathfrak{A}) \\ & \searrow^{B \mapsto \mathcal{G}_B \circ \mathcal{G}_A^{-1}} & \downarrow^{\mathcal{G}_B \mapsto \mathcal{G}_B \circ \mathcal{G}_A^{-1}} \\ & & \mathrm{C}(\mathrm{Sp}\, A) \end{array}$$

□

**Remark 2.4.** Let $\Phi : \mathrm{C}(\mathrm{Sp}\, A) \to \mathfrak{A}$ be the inverse of the isomorphism from the previous theorem defined by $B \mapsto \mathcal{G}_B \circ \mathcal{G}_A^{-1}$.

For $f \in \mathrm{C}(\mathrm{Sp}\, A)$, we get

$$\Phi(f) = \mathcal{G}^{-1}(f \circ \mathcal{G}_A)).$$

Thus, one retrieves the generators of $\mathfrak{A}$ via

$$\begin{aligned}\Phi(1_{\mathrm{Sp}(A)}) &= \mathcal{G}^{-1}(1_{\mathrm{Sp}(A)} \circ \mathcal{G}_A) \\ &= \mathcal{G}^{-1}(1_{\mathrm{Sp}(\mathfrak{A})}) \\ \Phi(\mathrm{id}_{\mathrm{Sp}(A)}) &= \mathcal{G}^{-1}(\mathrm{id}_{\mathrm{Sp}(A)} \circ \mathcal{G}_A) \\ &= \mathcal{G}^{-1}(\mathcal{G}_A) = A \\ \Phi(\overline{\mathrm{id}}_{\mathrm{Sp}(A)}) &= A^*.\end{aligned}$$

$\Phi$ gives us the possibility to identify functions on the closure of polynomials in $z, \overline{z}$ on $\mathrm{Sp}(A)$ with elements in $A$. By the theorem of Stone-Weierstrass, the closure of polynomials in $z, \overline{z}$ on $\mathrm{Sp}(A)$ are all continuous functions on $\mathrm{Sp}(A)$. Furthermore $\Phi$ is completely determined by its values on $1_{\mathrm{Sp}\, A}$ and $id_{\mathrm{Sp}\, A}$.

**Example 2.5.** Let $\mathfrak{B} = \mathcal{B}(\mathcal{H})$ be the space of bounded linear operators on some Hilbert space $\mathcal{H}$, $T$ a normal element and $\mathfrak{A}$ the $C^*$-algebra generated by $T$. Any entire function $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\mathrm{Sp}(A)$, and hence gives us an element $f(A)$ in $\mathfrak{A}$.

In general, a complex square root does not give a holomorphic function on $\mathrm{Sp}(A)$. However for self adjoint $A$, we can still define a continuous square root, as $\mathrm{Sp}(A)$ consists only of real numbers.

$$\sqrt{\ } : \mathrm{Sp}(A) \to \mathbb{C},\ z \mapsto \begin{cases} \sqrt{z} & \text{if } z > 0, \\ i\sqrt{-z} & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

If the normal operator $T$ is invertible, 0 is not an element of the spectrum. Since the spectrum is a closed subset of $\mathbb{C}$, there is a neighbourhood $U$ around 0, which is not contained in the spectrum of $T$. Thus $f(x) = 1/x$ is a continuous function on $\mathrm{Sp}(T)$. Since we have $1 = xf(x)$, the spectral theorem implies, that $f$ corresponds to $T^{-1}$.

**Proposition 2.6.** *Let $\mathfrak{B}$ be an involutive, unital Banach algebra, $\mathfrak{A}$ a unital $C^*$-algebra, and let*

$$\Phi : \mathfrak{B} \to \mathfrak{A}$$

*be an involutive algebra homomorphism. Then $\Phi$ is continuous and norm decreasing.*

*Proof.* Let $B \in \mathfrak{B}$. We have

$$\mathrm{Sp}_{\mathfrak{A}}(\Phi(B)) \subset \mathrm{Sp}_{\mathfrak{B}}(B).$$

For the spectral radius one has

$$\rho(\Phi(B)) \leq \rho(B) \leq \|B\|.$$

And consequently

$$\begin{aligned}\|\Phi(B)\|^2 &= \|(\Phi(B))^*\Phi(B)\| \\ &= \|\Phi(B^*B)\| \\ &= \rho(\Phi(B^*B)) \leq \|B^*B\| \leq \|B\|^2.\end{aligned}$$

This gives

$$\|\Phi\| \leq 1.$$

□

**Corollary 2.7.** *Using the same notation as before, the isomorphism*

$$\Phi : \mathrm{C}(\mathrm{Sp}\,\mathfrak{A}) \to \mathfrak{A}\ ,\ f \mapsto \mathcal{G}^{-1}(f \circ \mathcal{G}_A)$$

*is the only $C^*$-algebra homomorphism with the property that*

$$\Phi(1_{\mathrm{Sp}\,A}) = I \text{ and } \Phi(\mathrm{id}_{\mathrm{Sp}\,A}) = A.$$

*Proof.* If $\Psi : \mathrm{C}(\mathrm{Sp}\,A) \to \mathfrak{A}$ is another algebra homomorphism with the properties above, then $\Psi = \Phi$ on all polynomials in $z$ and $\overline{z}$ on $\mathrm{Sp}(A)$. We already know that both homomorphisms are continuous. Thus they must coincide on $\mathrm{C}(\mathrm{Sp}\,A)$ by the theorem of Stone-Weierstrass. □

# 3 Unbounded Operators

Since we have a functional calculus for normal bounded operators, one might hope that we can extend our results to unbounded operators. But the previous result relied on the Gelfand transform, which in turn relied on the existence of certain structures, such as the operator being an element in an algebra. But as previously remarked, closed operators are not that nice. This chapter follows [3]. Lemma 3.3 is taken from [1, p. 319].

From now on, let $T \in \mathscr{L}(\mathcal{H})$ be a closed normal operator. We endow $\mathfrak{D}(T)$ with the graph scalar product

$$\langle x, y \rangle_T := \langle \iota x, \iota y \rangle_{\mathcal{H}} + \langle T\iota x, T\iota y \rangle_{\mathcal{H}},$$

making it a Hilbert space. The topology given by the graph scalar product is finer than the subspace topology, as convergence in the graph norm implies convergence in the subspace topology. Furthermore, $T$ seen as a map from $(\mathfrak{D}(T), \langle \cdot, \cdot \rangle_T)$ to $\mathcal{H}$, is continous.

If there is no room for misinterpretation, we will omit the $\mathcal{H}$ in the scalar product. The adjoint of $T$ as a closed operator from $\mathcal{H}$ to itself, will be called $T^*$. Let $\iota : \mathfrak{D}(T) \to \mathcal{H}$ be the inclusion. We have two ways to interpret this map;

1. as an operator on $\mathcal{H}$, namely the identity with domain $\mathfrak{D}(T)$, or
2. as a bounded linear operator $\iota : (\mathfrak{D}(T), \langle \cdot, \cdot \rangle_T) \to (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$.

Using the second interpretation, we note that by $\mathfrak{R}(\iota^*) = \ker(\iota)^\perp = \mathcal{H}$ and $\ker(\iota^*) = \mathfrak{R}(\iota)^\perp = 0$, $\iota^*$ is a bijection, as $\mathfrak{D}(T)$ is dense in $\mathcal{H}$.

**Lemma 3.1.** *$\mathfrak{D}(T^*T)$ is dense in $(\mathfrak{D}(T), \langle \cdot, \cdot \rangle_T)$.*

*Proof.* We show that $\mathfrak{D}(T^*T)^{\perp_T} = 0$. Let $x \in \mathfrak{D}(T^*T),\ y \in \mathfrak{D}(T^*T)^{\perp_T}$. Then

$$\begin{aligned} 0 = \langle x, y \rangle_T &= \langle \iota x, \iota y \rangle_{\mathcal{H}} + \langle T\iota x, T\iota y \rangle_{\mathcal{H}} \\ &= \langle \iota x, \iota y \rangle_{\mathcal{H}} + \langle T^*T\iota x, \iota y \rangle_{\mathcal{H}} = \langle (I + T^*T)\iota x, \iota y \rangle_{\mathcal{H}}. \end{aligned}$$

Therefore, if we prove $\mathfrak{R}(I + T^*T)$ is dense in $\mathcal{H}$, the claim is proven as well. Since $\mathfrak{R}(I + T^*T)^{\perp_{\mathcal{H}}} = \ker(I + T^*T)$, we prove injectivity of $(I + T^*T) \in \mathscr{L}(\mathcal{H})$,

$$\begin{aligned} \|(I + T^*T)x\|^2 &= \|x\|^2 + \|T^*Tx\|^2 + \langle x, T^*Tx \rangle + \langle T^*Tx, x \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + 2\langle Tx, Tx \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + 2\|Tx\|^2 \geq \|x\|^2. \end{aligned}$$

□

**Proposition 3.2.** *$(I + T^*T)$ is boundedly invertible, with inverse $\iota\iota^*$.*

*Proof.* For $x \in \mathfrak{D}(T)$, $y \in \mathcal{H}$ such that $\iota\iota^* y \in \mathfrak{D}(T^*T)$. By definition

$$\begin{aligned} \langle \iota x, y \rangle_{\mathcal{H}} = \langle x, \iota^* y \rangle_T &= \langle \iota x, \iota\iota^* y \rangle_{\mathcal{H}} + \langle T\iota x, T\iota\iota^* y \rangle_{\mathcal{H}} \\ &= \langle \iota x, \iota\iota^* y \rangle_{\mathcal{H}} + \langle \iota x, T^*T\iota\iota^* y \rangle_{\mathcal{H}} = \langle \iota x, \iota\iota^* y + T^*T\iota\iota^* y \rangle_{\mathcal{H}}. \end{aligned}$$

Substracting the left hand side, we get

$$0 = \langle \iota x, \iota\iota^* y + T^*T\iota\iota^* y - y \rangle .$$

Note that since $\iota^*$ is a continuous bijection and $\mathfrak{D}(T^*T) \subset \mathfrak{D}(T)$ is a dense subset of both $\mathfrak{D}(T)$ and $\mathcal{H}$, $(i^*)^{-1}(\mathfrak{D}(T^*T))$ is dense in $\mathcal{H}$. Since this equality holds for all $x \in \mathfrak{D}(T)$, we get

$$y = \iota\iota^* y + T^*T\iota\iota^* y = (1 + T^*T)\iota\iota^* y,$$

which implies that $\iota\iota^* = (I + T^*T)^{-1}$, as $\iota\iota^*$ is bounded. □

Define $A \coloneqq \iota\iota^*$ and $B \coloneqq TA$. If we think of $A$ corresponding $1/1+|x|^2$, then we would expect $B$ to be bounded as well. As it turns out, this is true proven by the following

**Lemma 3.3.** *$B = TA = T(I + T^*T)^{-1}$ is a bounded operator, and we have $AT \subset TA$.*

*Proof.* Let $y \in \mathfrak{D}(I + T^*T)$ such that $(I + T^*T)y = x \in \mathfrak{D}(T)$. Using the calculation at the end of Lemma 3.1, we get $\|x + T^*Tx\|^2 \geq \|Tx\|^2$. From that, $\|TAx\|^2 = \|Ty\|^2 \leq \|(I + T^*T)y\|^2 = \|x\|^2$ , which proves that $B$ is bounded.

To show that $AT \subset TA$, take $y \in \mathfrak{D}(AT) = \mathfrak{D}(T)$, $x \in \mathfrak{D}(T^*T)$ such that $y = (I + T^*T)x$. $T^*Tx \in \mathfrak{D}(T)$ which implies $Tx \in \mathfrak{D}(TT^*) = \mathfrak{D}(T^*T)$. Then

$$ATy = A(Tx + TT^*Tx) = A((I + T^*T)Tx) = A(I + T^*T)Tx = Tx,$$

and

$$TAy = T(I + T^*T)^{-1}(I + T^*T)x = Tx,$$

concluding that $AT = TA$ on $\mathfrak{D}(T)$.

□

The operator $AT$ is bounded but not defined on all of $\mathcal{H}$. So we extend it in the following

**Lemma 3.4.** *$AT$ admits a bounded linear extension $\overline{AT}$ to all of $\mathcal{H}$. We then have $\overline{AT} = TA$.*

*Proof.* For $x \in \mathcal{H}$ we can choose a sequence $x_n \in \mathfrak{D}(T)$, with $x_n \to x$. Define $\overline{AT}(x) \coloneqq TA(x)$. This is linear, because $AT = TA$ on $\mathfrak{D}(T)$. As $TA$ is linear bounded, the limit does not depend on the chosen sequence. □

**Remark 3.5.** The previous two lemmata and their proofs, still hold if we replace $T$ by $T^*$, giving us

$$AT^* = T^*A \text{ and hence } B^* = T^*A.$$

One also has the identity

$$\begin{aligned} A^2 + B^*B &= (I + T^*T)^{-2} + T^*(I + T^*T)^{-1}T(I + T^*T)^{-1} \\ &= (I + T^*T)^{-2} + T^*T(I + T^*T)^{-1}(I + T^*T)^{-1} \\ &= (I + T^*T)(I + T^*T)^{-2} \\ &= (I + T^*T)^{-1} \\ &= A. \end{aligned}$$

From now on, we identify $B = \overline{AT}$ and $B^* = \overline{AT^*}$ with their bounded extensions. Define $\mathfrak{A} = \mathfrak{A}(T)$ by $\mathfrak{A} := \langle I, A, B, B^* \rangle$. Let $\chi \in \mathrm{Sp}(\mathfrak{A})$ , such that $\chi(A) = 0$. By our previous identity, we get

$$\chi(A)^2 + |\chi(B)|^2 = \chi(A),$$

which implies that $\chi(B) = 0$ as well. But for all $\chi$ in $\mathrm{Sp}(\mathfrak{A})$, it holds that $\chi(I) = 1$. If such a $\chi$ exists, it is therefore unique. We call this character $\chi_\infty$.

We define $\theta : \mathrm{Sp}(\mathfrak{A}) \to \overline{\mathbb{C}}$ by

$$\chi \mapsto \begin{cases} \frac{\chi(B)}{\chi(A)} & \text{, if } \chi \neq \chi_\infty \\ \infty & \text{, if } \chi = \chi_\infty. \end{cases}$$

Let $\chi \neq \chi_\infty$. Since $\chi$ is a involutive algebrahomomorphism, $A^2 + B^*B = A$ implies for $\chi(A) = \chi(A)\overline{\chi(A)} + \chi(B)\overline{\chi(B)}$ , which is equivalent to $\frac{1}{\chi(A)} = 1 + \frac{\chi(B)}{\chi(A)}\overline{\left(\frac{\chi(B)}{\chi(A)}\right)} = 1 + |\theta(\chi)|^2$. Inverting the last equality gives

$$\chi(A) = \frac{1}{1 + |\theta(\chi)|^2}. \tag{$*$}$$

The definition of $\theta$ (and not $T =$ "$\frac{B}{A}$"), gives

$$\chi(B) = \chi(A)\frac{\chi(B)}{\chi(A)} = \chi(A)\ \theta(\chi). \tag{$**$}$$

Recalling the definition of the Gelfandtransformation, we see that our map

$$\theta : \mathrm{Sp}(\mathfrak{A}) \setminus \{\chi_\infty\} \to \mathbb{C}$$

equals a fraction of two single Gelfandtransformations

$$\theta(\chi) = \frac{\chi(B)}{\chi(A)} = \frac{\mathcal{G}_B(\chi)}{\mathcal{G}_A(\chi)} = \frac{\mathcal{G}_B}{\mathcal{G}_A}(\chi).$$

But $\mathcal{G}_A \neq 0$ on $\mathrm{Sp}(\mathfrak{A})\setminus\{\chi_\infty\}$, which implies that $\theta$ is continuous on $\mathrm{Sp}(\mathfrak{A})\setminus\{\chi_\infty\}$. If $\chi_\infty$ exists, let $(\chi_\lambda)_{\lambda\in\Lambda}$ be a net converging to $\chi_\infty$ and $\chi_\lambda \neq \chi_\infty$ for all $\lambda \in \Lambda$. By continuity of $\mathcal{G}_A$ we have

$$\mathcal{G}_A(\chi_\lambda) = \chi_\lambda(A) \to \chi_\infty(A) = 0.$$

Equation $(*)$ implies

$$|\theta(\chi_\lambda)|^2 + 1 = \frac{1}{\chi(A)} \to \infty,$$

which is equivalent to

$$|\theta(\chi_\lambda)| \to \infty.$$

Summarizing the last part, we get

**Lemma 3.6.** *$\theta$ extends to a continuous map $n\theta : \mathrm{Sp}(\mathfrak{A}) \to \overline{\mathbb{C}}$, by $\theta(\chi_\infty) := \infty$. Furthermore $\theta : \mathrm{Sp}(\mathfrak{A}) \to \overline{\mathbb{C}}$ is a homeomorphism onto its image.*

*Proof.* The first claim was proven before. For the second claim we check $\theta$ is injective: Let $\chi_1, \chi_2 \neq \chi_\infty$. Equations $(*)$ and $(**)$ imply that, if $\theta(\chi_1) = \theta(\chi_2)$, $\chi_1$ coincides with $\chi_2$. Furthermore, $\chi_\infty$ is unique, which implies that $\theta$ is injective. Since $\mathrm{Sp}(\mathfrak{A})$ is compact and $\overline{\mathbb{C}}$ is Hausdorff, this proves the second claim. □

Combining the Gelfandisomorphism $\mathcal{G} : \mathfrak{A} \to \mathrm{C}(\mathrm{Sp}\,\mathfrak{A})$, with $\theta$, one has

$$\begin{gathered}\mathfrak{A} \longrightarrow \mathrm{C}(\,\mathrm{Sp}\,\mathfrak{A}) \longrightarrow \mathrm{C}(\theta(\mathrm{Sp}\,\mathfrak{A}))\\ x \mapsto \mathcal{G}_x \ ; \ f \mapsto f \circ \theta^{-1}\\ \mathcal{G}^{-1} f \leftarrow\!\shortmid f \ ; \ g \circ \theta \leftarrow\!\shortmid g.\end{gathered}$$

Define

$$\Phi : \mathrm{C}(\theta(\mathrm{Sp}\,\mathfrak{A})) \to \mathfrak{A}, \ g \mapsto \mathcal{G}^{-1}(g \circ \theta).$$

As $\mathcal{G}$ is an involutive algebrahomomorphism, $\Phi$ is as well.

**Proposition 3.7.** *Let $T$ be a normal operator on $\mathcal{H}$, $\mathfrak{A} = \langle I, A, B \rangle$ the $C^*$-algebra associated to $T$. Then $\Phi$ is the only involutive algebrahomomorphism from $\mathrm{C}(\theta(\mathrm{Sp}\,\mathfrak{A}))$ onto $\mathfrak{A}$, such that*

$$\Phi\left(\frac{1}{1+|z|^2}\right) = A, \ \Phi\left(\frac{z}{1+|z|^2}\right) = B.$$

*Proof.*

$$\begin{aligned}\Phi^{-1}(A)(z) &= \mathcal{G}_A \circ \theta^{-1}(z)\\ &= \mathcal{G}_A(\theta^{-1}(z))\\ &= \frac{1}{1+|z|^2}\end{aligned}$$

using $\mathcal{G}_A(\chi) = \chi(A) = \frac{1}{1+|\theta(\chi)|^2}$. Moreover

$$\begin{aligned}\Phi^{-1}(B)(z) &= \mathcal{G}_B \circ \theta^{-1}(z)\\ &= \mathcal{G}_B(\theta^{-1}(z))\\ &= \frac{z}{1+|z|^2}.\end{aligned}$$

To proof uniqueness, let $\Psi : \mathrm{C}(\theta(\mathrm{Sp}\,\mathfrak{A})) \to \mathfrak{A}$ be another involutive algebrahomomorphism such that

$$\Psi\left(\frac{1}{1+|z|^2}\right) = A, \ \Psi\left(\frac{z}{1+|z|^2}\right) = B.$$

Then $\Phi$ coincides with $\Psi$ on all polynomials in $A, B, \overline{B}$. The polynomials form an involutive algebra, which seperates points. By the theorem of Stone-Weierstrass it is dense, and therefore $\Phi$ equals $\Psi$ since they are continious. □

Our goal is to construct a functional calculus for $T$. With respect to $\Phi$, $T$ corresponds to $I_{\theta(\mathrm{Sp}\,\mathfrak{A})}$. But if $\chi_\infty \in \mathrm{Sp}(\mathfrak{A})$, then $I_{\theta(\mathrm{Sp}\,\mathfrak{A}))} \notin \mathrm{C}(\mathrm{Sp}\,\mathfrak{A})$ since $\infty \notin \mathbb{C}$.

# 4 Extension of Continuous Spectral Measures

This section will follow [3] and [5, pp. 341-] , while some notation is taken from [4, Ch. 6]. The reader not familiar with complex measures, is advised to review the basic facts in [6, Ch. 6].

We want to extend a given spectral measure for continuous functions, like the one we obtained in Proposition 2.3, to measurable ones. To do so, we need to check whether the operators we attain are bounded, or densely defined.

**Definition 4.1** (Spectral measure)**.** Let $X$ be a compact space and let

$$\Phi : \mathrm{C}(X) \to \mathscr{L}(\mathcal{H})$$

be a map. $\Phi$ is called a *spectral measure*, if its image $\mathfrak{A} := \Phi(\mathrm{C}(X))$ is a commutative star-subalgebra of $\mathscr{L}(\mathcal{H})$ and $\Phi$ induces an isomorphism onto its image.

**Remark 4.2.** By isomorphism we mean that

1. $\Phi$ is an involutive algebra homomorphism,
2. $\Phi$ is a bijection onto A,
3. $\Phi$ is an isometry: $\|\Phi f\| = \|f\|_\infty$.

**Example 4.3.** Let $\mathfrak{m}$ be a positive Radon integral on our compact space $X$, meaning a continuous linear functional on $\mathrm{C}(X)$, such that $\mathfrak{m}(f) = 0$ implies that $f = 0$. An more in depth study of Radon integrals can be found in [4, Chapter 6.1]. Set $\mathcal{H} := L^2(\mathfrak{m})$. We define

$$\begin{aligned} \Phi : \mathrm{C}(X) &\to \mathscr{L}(L^2(m)) \\ f &\mapsto (g \mapsto f \cdot g). \end{aligned}$$

$\Phi$ is a spectral measure. This claim is not obvious, and the proof that $\Phi$ is an isometry requires a bit of work, but is omitted here. It can be found in [3].

Departing from this example, let $\mathcal{H}$ be a Hilbert space. For all $g, h \in \mathcal{H}$ define

$$\mathfrak{m}_{g,h}(f) := \langle g, \Phi_f h \rangle .$$

We then conclude, that the map

$$f \mapsto \mathfrak{m}_{g,h}(f)$$

is a linear form on $\mathrm{C}(X)$ for every pair $(g, h) \in \mathcal{H} \times \mathcal{H}$.

**Theorem 4.4.** *For all $g, h, k \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and $f, \phi, \psi \in \mathrm{C}(X)$, it holds that $\mathfrak{m}_{g,h}$ is a Radon integral on $X$ with the following properties:*

*(i)* $\|\mathfrak{m}_{g,h}\| \leq \|g\|\|h\|$

*(ii)* $\mathfrak{m}_{\alpha g+h,\beta k} = \overline{\alpha}\beta\mathfrak{m}_{g,k} + \beta\mathfrak{m}_{h,k}$

*(iii)* $\overline{\mathfrak{m}}_{g,h} = \mathfrak{m}_{h,g}$, $\overline{\mathfrak{m}}_{g,h} : f \mapsto \overline{\mathfrak{m}}_{g,h}(f) = \overline{\mathfrak{m}_{g,h}(\overline{f})}$

*(iv)* $\mathfrak{m}_{g,g} \geq 0$

*(v)* $\mathfrak{m}_{\Phi_\phi g, \Phi_\psi h} = \overline{\phi}\psi \mathfrak{m}_{g,h}$.

*Proof.* (i)

$$\begin{aligned} \|\mathfrak{m}_{g,h}\| &= \sup_{\|f\|_\infty \le 1} |\mathfrak{m}_{g,h}(f)| \\ &= \sup_{\|f\|_\infty \le 1} |\langle g, \Phi_f h\rangle| \\ &\le \sup_{\|f\|_\infty \le 1} \|g\|\|\Phi_f\|\|h\| \\ &= \|g\|\|h\| \end{aligned}$$

since $\|\Phi_f\| = \|f\|_\infty$.

(ii) follows immediately from the linearity of $\langle \cdot, \cdot \rangle$

(iii) $\overline{\mathfrak{m}}_{g,h}(f) = \overline{\langle g, \overline{f} h\rangle} = \langle \overline{f} h, g\rangle = f \langle h, g\rangle = \langle h, fg\rangle = \mathfrak{m}_{h,g}(f)$.

(iv) Let $\phi \ge 0$. Because $\Phi$ is algebra homomorphism, and $\overline{\sqrt{\phi}} = \sqrt{\phi}$, we have $\Phi_\phi = \Phi_{\sqrt{\phi}}\Phi_{\sqrt{\phi}} = \Phi^*_{\sqrt{\phi}}\Phi_{\sqrt{\phi}}$. Therefore

$$\mathfrak{m}_{g,g}(\phi) = \langle g, \Phi_\phi g\rangle = \left\langle \Phi_{\sqrt{\phi}} g, \Phi_{\sqrt{\phi}} g\right\rangle \ge 0.$$

(v) $\mathfrak{m}_{\Phi_\phi g, \Phi_\psi h}(f) = \langle \Phi_\phi g, \Phi_f \Phi_\psi h\rangle = \left\langle g, \Phi_{\overline{\phi}\psi f} h\right\rangle = (\overline{\phi}\psi \mathfrak{m}_{g,h})(f)$ .

□

We now extend $\mathfrak{m}_{g,h}$ to measurable functions in the sense of [4, Ch. 4.5]. The definition of measurability without using $\sigma$-algebras, can be found in [4, Ch. 6.3].

**Definition 4.5.** $N \subset X$ is called a $\Phi$-set of measure zero, or $\Phi$-null set, if $N$ is a null set of $|\mathfrak{m}_{g,h}|$ for all $g, h \in \mathcal{H}$, that is $|\mathfrak{m}_{g,h}|(N) = 0$.

Note that $|\mathfrak{m}_{g,h}|$ is a real valued Radon integral. A function $f : X \to \mathbb{C}$ is called $\Phi$-measurable, if $f$ is $|\mathfrak{m}_{g,h}|$-measurable for all $g, h \in \mathcal{H}$. Denote the set of all measurable functions $\mathrm{L}^0(\Phi)$.

$$\mathrm{L}^1(\Phi) := \left\{ f \in \mathrm{L}^0(\Phi) \;\middle|\; f \in \mathrm{L}^1(\mathfrak{m}_{g,h}) \text{ for all } g, h \in \mathcal{H} \right\}$$
$$\mathrm{L}^\infty(\Phi) := \left\{ f \in \mathrm{L}^0(\Phi) \;\middle|\; \|f\|_\infty < \infty \right\}$$

where,

$$\|f\|_\infty := \inf\left\{ \lambda > 0 \;\middle|\; |f| \le \lambda,\ \Phi\text{-a.e.} \right\}.$$

By $\mathscr{E}(X)$ we denote the measurable subsets of $X$:

$$\mathscr{E}(X) := \left\{ A \subset X \;\middle|\; 1_A \in \mathrm{L}^0(\Phi) \right\}.$$

Let $f \in \mathrm{L}^0(\Phi)$.

$$\begin{aligned} \mathfrak{D}(f) := \Big\{ h \in \mathcal{H} \;\Big|\; & f \in \mathrm{L}^1(\mathfrak{m}_{g,h}) \text{ for all } g \in \mathcal{H} \\ & \text{and } g \mapsto \int f \,\mathrm{d}\mathfrak{m}_{g,h} \text{ is continuous} \Big\} \\ = \Big\{ h \in \mathcal{H} \;\Big|\; & f \in \mathrm{L}^1(\mathfrak{m}_{g,h}) \text{ for all } g \in \mathcal{H} \\ & \text{and } \exists k \in \mathcal{H} \text{ such that } \int f \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, k\rangle \Big\} \end{aligned}$$

where the second equality is due to the Riesz representation theorem. The reader familiar with unbounded operators, will recognize the similarity with the definition of the adjoint operator. Using the second equality, we define

$$\Phi_f h := k(h,f) = k \text{ for } h \in \mathfrak{D}(f) =: \mathfrak{D}(\Phi_f).$$

In other words we have

$$\int f \, \mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_f h \rangle = \langle g, k \rangle .$$

By sesquilinearity of $\langle \cdot, \cdot \rangle$, $\Phi_f$ is linear as well.

**Remark 4.6.** If we want to proof claims about $\mathrm{L}^0(\Phi)$ it is enough to show them for $\mathrm{C}(X)$, as every measurable function is the limit of continuous ones [4, Proposition 6.2.9].

**Lemma 4.7.** *For all $f \in L^0(\Phi)$ and $h \in \mathfrak{D}(f)$, $g \in \mathcal{H}$ it holds that*

$$\mathfrak{m}_{g,\Phi_f h} = f\mathfrak{m}_{g,h} \text{ and } \mathfrak{m}_{\Phi_f g,h} = \overline{f}\mathfrak{m}_{g,h}.$$

*Proof.* By the remark, let $\phi \in \mathrm{C}(X)$. Since $\Phi$ is a spectral measure, we get $\Phi_\phi^* = \Phi_{\overline{\phi}}$. Thus,

$$\begin{aligned} \mathfrak{m}_{g,\Phi_f h}(\phi) &= \langle g, \Phi_\phi(\Phi_f h) \rangle = \left\langle \Phi_{\overline{\phi}} g, \Phi_f h \right\rangle = \int f \, \mathrm{d}\mathfrak{m}_{\Phi_{\overline{\phi}} g,h} \\ &= \int f \, \mathrm{d}\mathfrak{m}_{g,\Phi_\phi h} = \int f\phi \, \mathrm{d}\mathfrak{m}_{g,h} = (f\mathfrak{m}_{g,h})(\phi). \end{aligned}$$

Since $h$ being in the domain of $f$ means, that for all $g \in \mathcal{H}$, $f$ is an element of $\mathrm{L}^1(\mathfrak{m}_{g,h})$, the expression $f\mathfrak{m}_{g,h}$ makes sense.

On the other hand, let $g \in \mathfrak{D}(f)$. We calculate

$$\begin{aligned} \mathfrak{m}_{\Phi_f g,h}(\phi) &= \langle \Phi_f g, \Phi_\phi h \rangle = \overline{\langle \Phi_\phi h, \Phi_f g \rangle} = \overline{\left\langle h, \Phi_{\overline{\phi}} \Phi_f g \right\rangle} \\ &= \overline{\int \overline{\phi} \, \mathrm{d}\mathfrak{m}_{h,\Phi_f g}} = \overline{\int \overline{\phi} f \, \mathrm{d}\mathfrak{m}_{h,g}} = \overline{f\mathfrak{m}_{h,g}(\overline{\phi})} \\ &= \overline{\mathfrak{m}_{h,g}(\overline{\phi} f)} = \overline{\mathfrak{m}}_{h,g}(\phi \overline{f}) = \mathfrak{m}_{g,h}(\phi \overline{f}) = (\overline{f}\mathfrak{m}_{g,h})(\phi). \end{aligned}$$

□

We now want to know, whether the operator we obtain from our expanded measure is bounded, or at least densely defined, if not bounded. This will be done in the next lemmata, which are going to be summarized in a theorem at the end of the chapter.

**Lemma 4.8.** *If $f \in L^\infty(\Phi)$, then $\mathfrak{D}(f) = \mathcal{H}$, $\Phi_f \in \mathscr{L}(\mathcal{H})$ and*

$$\|\Phi_f\| = \|f\|_\infty = \inf \left\{ \alpha > 0 \;\middle|\; \alpha \geq |f|, \ \Phi\text{-}a.e. \right\}.$$

*Proof.* Let $f \in \mathrm{L}^\infty(\Phi)$. We have to show that $\mathfrak{D}(f) = \mathcal{H}$. Let $h \in \mathcal{H}$. For $g \in \mathcal{H}$ one gets:

$$\begin{aligned}\left| \int f \, \mathrm{d}\mathfrak{m}_{g,h} \right| &\leq \int |f| \mathrm{d}|\mathfrak{m}_{g,h}| \\ &\leq \|f\|_\infty \int \mathrm{d}|\mathfrak{m}_{g,h}| \\ &= \|f\|_\infty \|\mathfrak{m}_{g,h}\| \\ &\leq \|f\|_\infty \|h\| \|g\|\end{aligned}$$

which implies

$$g \mapsto \int f \, \mathrm{d}\mathfrak{m}_{g,h}$$

is continuous and $f$ an element of $\mathscr{L}^1(\mathfrak{m}_{g,h})$. The third equality follows from

$$\|\mathfrak{m}_{g,h}\| = \sup_{\|\phi\|_\infty \leq 1} |\mathfrak{m}_{g,h}| = \sup_{\substack{|\phi| \leq 1 \\ \phi \in \mathrm{C}(X)}} |\mathfrak{m}_{g,h}| = \int \mathrm{d}|\mathfrak{m}_{g,h}| = |\mathfrak{m}_{g,h}|(1).$$

This shows that the assignment $g \mapsto \int f \, \mathrm{d}\mathfrak{m}_{g,h}$ defines a continuous map.

$$\|\Phi_f\| = \sup_{\|g\|, \|h\| \leq 1} |\langle g, \Phi_f h \rangle| = \sup_{\|g\|, \|h\| \leq 1} \left| \int f \, \mathrm{d}\mathfrak{m}_{g,h} \right| \leq \|f\|_\infty.$$

The other inequality, will be proved in Lemma 4.14. □

**Remark 4.9.** If $f_n \to f$ in $\mathrm{L}^\infty(\Phi)$, then

$$\Phi_f = \lim \Phi_{f_n} \text{ in } \mathscr{L}(\mathcal{H}).$$

By $\int^* f \, \mathrm{d}\mathfrak{m}_{g,h}$, we mean the upper integral, as defined in [4, Ch. 6.1]. If $\int^* f \, \mathrm{d}\mathfrak{m}_{g,h} < \infty$, then $\int f \, \mathrm{d}\mathfrak{m}_{g,h} = \int^* f \, \mathrm{d}\mathfrak{m}_{g,h}$, also found in [4].

**Lemma 4.10.** *Let $f \in L^0(\Phi), (f_n)$ a net in $L^\infty(\Phi)$, and $\alpha, \beta \geq 0$ such that*

$$\begin{aligned}|f_n| &\leq \alpha |f| + \beta \text{ for all } n \\ f_n &\to f \ \Phi\text{-a.e.}.\end{aligned}$$

*Then the following statements about $h \in \mathcal{H}$ are equivalent:*

*(i)* $h \in \mathfrak{D}(f)$

*(ii)* $\int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} < \infty$

*(iii)* $(\Phi_{f_n} h)$ *converges in* $\mathcal{H}$

*One then has*

$$\begin{aligned}\|\Phi_f h\|^2 &= \int |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} \text{ and} \\ \Phi_f h &= \lim \Phi_{f_n} h.\end{aligned}$$

*For example, one can take the net $f_n = 1_{\mathcal{A}_n}$, where $\mathcal{A}_n = \{x \in X \mid |f(x)| \leq n\}$.*

*Proof.* (*i*) ⇒ (*ii*): Let $h \in \mathfrak{D}(f)$.

$$\begin{aligned}\infty > \|\Phi_f h\|^2 &= \langle \Phi_f h, \Phi_f h \rangle \\ &= \int f \, \mathrm{d}\mathfrak{m}_{\Phi_f h, h} \\ &= \int f \overline{f} \, \mathrm{d}\mathfrak{m}_{h,h} \\ &= \int |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h}.\end{aligned}$$

(*ii*) ⇒ (*iii*): Let $h \in \mathscr{L}^2(\mathfrak{m}_{h,h})$. Then for $g \in \mathcal{H}$

$$\begin{aligned}\langle g, (\Phi_{f_m} - \Phi_{f_n})h \rangle &= \langle g, \Phi_{f_m} \rangle - \langle g, \Phi_{f_n} \rangle \\ &= \int f_m \, \mathrm{d}\mathfrak{m}_{g,h} - \int f_n \, \mathrm{d}\mathfrak{m}_{g,h} \\ &= \int (f_m - f_n) \, \mathrm{d}\mathfrak{m}_{g,h} \\ &= \left\langle g, \Phi_{(f_m - f_n)} h \right\rangle\end{aligned}$$

Since $g$ was arbitrary, we get $(\Phi_{f_m} - \Phi_{f_n})h = \Phi_{(f_m - f_n)} h$

$$\begin{aligned}\|(\Phi_{f_m} - \Phi_{f_n})h\|^2 &= \|\Phi_{(f_m - f_n)} h\|^2 \\ &= \int |f_m - f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h}.\end{aligned}$$

where the last equality holds, because $h \in \mathfrak{D}(f_m - f_n)$ and thus $f \in \mathrm{L}^2(\Phi)$. By our assumption, we have $f_m - f_n \to 0$ $\Phi$-a.e. Therefore

$$|f_m - f_n|^2 \leq (2(\alpha|f| + \beta))^2 \in \mathscr{L}^1(\mathfrak{m}_{h,h}),$$

as constant functions are contained $\mathrm{L}^1(\Phi)$. Now, Lebegues theorem about dominated convergence yields the claim.

(*iii*) ⇒ (*ii*): Using Fatous lemma, and $\|\Phi_{f_n} h\|^2 = \int |f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h}$, we get

$$\begin{aligned}\infty > \|\lim_{n\to\infty} \Phi_{f_n} h\|^2 &= \lim_{n\to\infty} \|\Phi_{f_n} h\|^2 = \lim_{n\to\infty} \int |f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h} \\ &\geq \int^* \liminf_{n\to\infty} |f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h} = \int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h}\end{aligned}$$

(*ii*) ⇒ (*i*): By the theorem of Radon–Nikodym, there exists a Borel measurable function $\phi : X \to \mathbb{C}$, $|\phi| = 1$, such that

$$|\mathfrak{m}_{g,h}| = \phi \mathfrak{m}_{g,h}.$$

Now define $\widetilde{f} \coloneqq \phi|f|$, $\widetilde{f_n} \coloneqq \phi|f_n|$. Thus

$$\int^* |\widetilde{f}|^2 \, \mathrm{d}\mathfrak{m}_{h,h} = \int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} < \infty.$$

As shown in (*ii*) ⇒ (*iii*), the limit of $\Phi_{\widetilde{f_n}}$ exists. Hence, for $g \in \mathcal{H}$

$$\infty > \|g\|^2 \| \lim_{n\to\infty} \Phi_{\widetilde{f}_n} h\|^2 \geq \left|\left\langle g, \lim_{n\to\infty} \Phi_{\widetilde{f}_n} h \right\rangle\right| = \left|\lim_{n\to\infty} \int \widetilde{f}_n \,\mathrm{d}\mathfrak{m}_{g,h}\right|$$
$$\geq \lim_{n\to\infty} \int |f_n| \mathrm{d}|\mathfrak{m}_{g,h}| \geq \int^* \liminf_{n\to\infty} |f_n| \mathrm{d}|\mathfrak{m}_{g,h}| = \int^* |f| \mathrm{d}|\mathfrak{m}_{g,h}|,$$

which implies that $f \in \mathscr{L}^1(\mathfrak{m}_{g,h})$. So once more by Lebegues theorem

$$\left\langle g, \lim_{n\to\infty} \Phi_{f_n} h \right\rangle = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mathfrak{m}_{g,h} = \int f \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_f h\rangle,$$

which means, $h \in \mathfrak{D}(f)$.

□

**Lemma 4.11.** *For each $f \in L^0(\Phi)$, $\Phi_f$ is a normal Operator, and we have*

$$\Phi_f^* = \Phi_{\overline{f}}.$$

*If $f$ is real valued, then $\Phi_f$ is self-adjoint, and $\Phi_{1_A} =: E_A$ for $A \in \mathscr{E}(X)$ is an orthogonal projection.*

*Proof.* First we show that $\Phi_f$ is densely defined. We claim $E_{\mathcal{A}_n}(\mathcal{H}) \subset \mathfrak{D}(f)$, where $\mathcal{A}_n = \{x \in X \mid |f(x)| \leq n\}$. Let $h \in \mathcal{H}$. By Lemma 4.10 the claim is equivalent to

$$\int^* |f|^2 \,\mathrm{d}\mathfrak{m}_{E_{\mathcal{A}_n}h, E_{\mathcal{A}_n}h} < \infty.$$

We have

$$\int^* |f|^2 \,\mathrm{d}\mathfrak{m}_{E_{\mathcal{A}_n}h, E_{\mathcal{A}_n}h} = \int^* \overline{1}_{\mathcal{A}_n} 1_{\mathcal{A}_n} \,\mathrm{d}\mathfrak{m}_{h,h} \leq n^2 \int \mathrm{d}\mathfrak{m}_{h,h} \leq n^2 \|h\|^2 < \infty.$$

For $h \in \mathcal{H}$ we have $h = \lim_{n\to\infty} E_{\mathcal{A}_n} h$. Then $|1_{\mathcal{A}_n}| \leq 1$, and we have $1_{\mathcal{A}_n} \to 1$ pointwise $\Phi$-a.e. Let $h \in \mathcal{H} = \mathfrak{D}(1)$. By Lemma 4.10 we have

$$h = \Phi_1 h = \lim_{n\to\infty} \Phi_{1_{\mathcal{A}_n}} = \lim_{n\to\infty} E_{\mathcal{A}_n} h,$$

which gives $\mathfrak{D}(f) \subset \mathcal{H}$ is dense, as $E_{\mathcal{A}_n} h \in \mathfrak{D}(f)$ by the claim.

Now, we claim $\Phi_{\overline{f}} \subset \Phi_f^*$. Let $g, h \in \mathfrak{D}(f) = \mathfrak{D}(\overline{f})$. Using Theorem 4.4 (*iii*), one has

$$\langle g, \Phi_f h\rangle = \int f \,\mathrm{d}\mathfrak{m}_{g,h} = \overline{\int \overline{f} \mathrm{d}\overline{\mathfrak{m}_{g,h}}} = \overline{\left\langle h, \Phi_{\overline{f}} g \right\rangle} = \left\langle \Phi_{\overline{f}} g, h \right\rangle$$

Thus, $g \in \mathfrak{D}(\Phi_f^*)$ and $\Phi_f^* g = \Phi_{\overline{f}} g$ for all $g \in \mathfrak{D}(f)$.

On the other hand, to show that $\Phi_{\overline{f}} \supset \Phi_f^*$ let $g \in \mathfrak{D}(\Phi_f^*)$. By Lemma 4.10, we only have to show $\Phi_{\overline{f}_n} g$ converges in $\mathcal{H}$, for some net satisfyig the conditions of Lemma 4.10. As a net, we take $f_n = f \cdot 1_{\mathcal{A}_n}$, where $\mathcal{A}_n = \{x \in X \mid |f(x)| \leq n\}$. Let $h \in \mathcal{H}$. For better readability, we write $E_n$ for $\Phi_{\mathcal{A}_n}$, $F_n$, $F_n^*$ for $\Phi_{f_n}$ respectively $\Phi_{\overline{f}_n}$, and $F$ for $\Phi_f$.

$$\begin{aligned}\langle F_n^* g, h\rangle &= \int \mathrm{d}\mathfrak{m}_{F_n^* g,h} = \int f_n \,\mathrm{d}\mathfrak{m}_{g,h} = \int f \,\mathrm{d}\mathfrak{m}_{g,E_n h} = \int \mathrm{d}\mathfrak{m}_{g,FE_n h} \\ &= \langle g, FE_n h\rangle = \langle F^* g, E_n h\rangle = \int 1_{\mathcal{A}_n} \,\mathrm{d}\mathfrak{m}_{F^* g,h} \\ &= \int \mathrm{d}\mathfrak{m}_{E_n F^* g,h} = \langle E_{\mathcal{A}_n} F^* g, h\rangle\end{aligned}$$

Now $\Phi_{\overline{f}_n} g = E_{\mathcal{A}_n} \Phi_f^* g \xrightarrow{n\to\infty} \Phi_f^* g$, as $E_{\mathcal{A}_n}$ converges to the identity. It follows, that $\Phi_{\overline{f}} \supset \Phi_f^*$, completing the proof of $\Phi_{\overline{f}} = \Phi_f^*$.

Next, we claim $\Phi_f$ is a normal element. We have to show, that $\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_f^*)$ and $\|\Phi_f^* h\| = \|\Phi_f h\|$. The first claim is already proven, because $\mathfrak{D}(f) = \mathfrak{D}(\overline{f})$. The second condition is also fulfilled, which can be seen by the norm formula of Lemma 4.10. Thus $\mathfrak{D}(\Phi_f^*) = \mathfrak{D}(\Phi_{\overline{f}})$ and $\|\Phi_f^* h\| = \|\Phi_f h\|$, which proves that $\Phi_f$ is normal. To show that this implies the usual definition of a normal operator, i.e. $\Phi_f \Phi_f^* = \Phi_f^* \Phi_f$, can easily be seen, using the polarization identity, found in [7, Ch. 4.6].

If $f$ is real valued, we have that $f = \overline{f}$, which gives the selfadjointness of $\Phi_f$. Furthermore $E_{\mathcal{A}}^* = E_{\mathcal{A}}$. As $(E_{\mathcal{A}})^2 = E_{\mathcal{A}}$, we get that $E_{\mathcal{A}}$ is an orthogonal projection.

□

**Corollary 4.12.**

1. $f \in L^\infty(\Phi)$,
$$f \geq 0 \ \Phi\text{-}a.e. \Rightarrow \Phi_f \geq 0$$

2. $\mathcal{A} \in \mathscr{E}(X)$,
$$E_{\mathcal{A}} = 0 \Leftrightarrow \mathcal{A} \ \Phi\text{-}null\ set$$

3. $U \in \mathscr{E}(X)$ *open,*
$$U \neq \varnothing \Rightarrow E_U \neq 0$$

*Proof.* Immediate consequence of the previous lemmata. Full proof found in [3]. □

**Lemma 4.13.** *For each* $\varphi, \psi \in L^0(\Phi)$, $\alpha \in \mathbb{C}$, *we have*

*(i)* $\Phi_{\alpha\varphi} = \alpha\Phi_\varphi$

*(ii)* $\mathfrak{D}(\Phi_\varphi \Phi_\psi) = \mathfrak{D}(\varphi\psi) \cap \mathfrak{D}(\psi)$, *and* $\overline{\Phi_\varphi \Phi_\psi} = \Phi_{\varphi\psi}$

*(iii)* $\overline{\Phi_\varphi + \Phi_\psi} = \Phi_{\varphi+\psi}$

*(iv)* $\psi \in L^\infty(\Phi) \Rightarrow \Phi_\varphi + \Phi_\psi = \Phi_{\varphi+\psi}$, *and* $\Phi_\varphi \Phi_\psi = \Phi_{\varphi\psi}$.

*Proof.* $(i)$: By definition, we get

$$\mathfrak{D}(\alpha\varphi) = \left\{ h \in \mathcal{H} \;\Big|\; \int^* |\alpha\varphi|^2 \,\mathrm{d}\mathfrak{m}_{h,h} < \infty \right\} = \mathfrak{D}(\varphi),$$

and

$$\langle g, \Phi_{\alpha\varphi} h \rangle = \int \alpha\varphi \,\mathrm{d}\mathfrak{m}_{g,h} = \alpha \int \varphi \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \alpha\Phi_\varphi h \rangle\,,$$

for all $g \in \mathcal{H}$ and $h \in \mathfrak{D}(\phi)$.

(*ii*): $h \in \mathfrak{D}(\Phi_\varphi \Phi_\psi)$ reformulates to $h \in \mathfrak{D}(\psi)$ and $\Phi_\psi h \in \mathfrak{D}(\varphi)$. By Lemma 4.7, $\mathfrak{m}_{g,\Phi_\psi h} = \psi \mathfrak{m}_{g,h}$ for $h \in \mathfrak{D}(\psi)$. We compute

$$\int^* |\varphi| \mathrm{d} |\mathfrak{m}_{g,\Phi_\psi h}| = \int^* |\varphi\psi| \mathrm{d} |\mathfrak{m}_{g,h}|,$$

so

$$\int^* |\varphi| \mathrm{d} |\mathfrak{m}_{g,\Phi_\psi h}| < \infty \text{ is equivalent to } \int^* |\varphi\psi| \mathrm{d} |\mathfrak{m}_{g,h}| < \infty.$$

Therefore,

$$g \mapsto \int^* \varphi \,\mathrm{d}\mathfrak{m}_{g,\Phi_\psi h} \text{ is continuous } \Leftrightarrow g \mapsto \int^* \varphi\psi \,\mathrm{d}\mathfrak{m}_{g,h} \text{ is continuous.}$$

The last statement, reformulates to $h \in \mathfrak{D}(\psi)$ and $\Phi_\psi h \in \mathfrak{D}(\varphi)$ which is equivalent to $h \in \mathfrak{D}(\psi)$ and $h \in \mathfrak{D}(\varphi\psi)$.
As $\langle g, \Phi_\varphi \Phi_\psi h \rangle = \int \varphi \,\mathrm{d}\mathfrak{m}_{g,\Phi_\psi h} = \int \varphi\psi \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_{\varphi\psi} h \rangle$ , we get $\Phi_\varphi \Phi_\psi \subset \Phi_{\varphi\psi}$. The proof of $\overline{\Phi_\varphi \Phi_\psi} = \Phi_{\varphi\psi}$ is analogous to the proof of $\overline{\Phi_\varphi + \Phi_\psi} = \Phi_{\varphi+\psi}$ and will be omitted.

(*iii*): To show $\mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi) \subset \mathfrak{D}(\varphi + \psi)$. Let $h \in \mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi)$. By Lemma 4.10,

$$\int^* |\varphi|^2 \,\mathrm{d}\mathfrak{m}_{h,h},\ \int^* |\psi|^2 \,\mathrm{d}\mathfrak{m}_{h,h} < \infty,$$

and by Minkowskys inequality

$$\left(\int^* |\varphi + \psi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}\right)^{\frac{1}{2}} \leq \left(\int^* |\varphi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}\right)^{\frac{1}{2}} + \left(\int^* |\psi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}\right)^{\frac{1}{2}}.$$

Thus $h \in \mathfrak{D}(\varphi + \psi)$. Furthermore, for $g \in \mathcal{H}$

$$\langle g, (\Phi_\varphi + \Phi_\psi) h \rangle = \int \varphi \,\mathrm{d}\mathfrak{m}_{g,h} + \int \psi \,\mathrm{d}\mathfrak{m}_{g,h} = \int (\varphi + \psi) \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_{\varphi+\psi} h \rangle\,,$$

and thus $\Phi_\varphi + \Phi_\psi \subset \Phi_{\varphi+\psi}$. The rest of the proof will follow after part (*iv*).

(*iv*): $\psi \in \mathrm{L}^\infty(\Phi)$ implies that $\mathfrak{D}(\psi)$ is already the whole space. Thus

$$\mathfrak{D}(\Phi_\varphi \Phi_\psi) = \mathfrak{D}(\varphi\psi) \cap \mathcal{H} = \mathfrak{D}(\varphi\psi),$$

and

$$\mathfrak{D}(\Phi_\varphi + \Phi_\psi) = \mathfrak{D}(\varphi) \cap \mathcal{H} = \mathfrak{D}(\varphi).$$

Therefore, we have proven

$$\Phi_\varphi \Phi_\psi = \Phi_{\varphi\psi} \text{ and } \Phi_\varphi \Phi_\psi = \Phi_{\varphi+\psi}.$$

In particular,

$$\mathcal{A} \in \mathscr{E}(\Phi) \Rightarrow E_\mathcal{A} \text{ is a projection.}$$

Rest of (*ii*): For $\overline{\Phi_\varphi + \Phi_\psi} = \Phi_{\varphi+\psi}$, we need to show that for $h \in \mathfrak{D}(\varphi + \psi)$, there exists a net $(h_n) \in \mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi)$, such that $\lim h_n = h$, and $\lim(\Phi_\varphi h_n +$

$\Phi_\psi h_n) = \Phi_{\varphi+\psi} h$. Set $\mathcal{A}_n = \{x \in X \mid |\varphi(x)| + |\psi(x)| \leq n\}$. By Lemma 4.10, we have

$$E_{\mathcal{A}_n}(\mathcal{H}) \subset \mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi)$$

and

$$\cup \mathcal{A}_n = X,\ \mathcal{A}_n \subset \mathcal{A}_{n+1}.$$

And thus

$$\lim E_{\mathcal{A}_n} = \mathrm{id}$$

For all $h \in \mathfrak{D}(\varphi + \psi)$, one has

$$h = \lim E_{\mathcal{A}_n} h =: \lim h_n,$$

and using $(iv)$ combined with the fact that $E_{\mathcal{A}_n}$ is bounded, we get

$$\begin{aligned}
\Phi_{\varphi+\psi} h &= \lim E_{\mathcal{A}_n}(\Phi_{\varphi+\psi})h \\
&= \lim \Phi_{1_{\mathcal{A}_n}(\varphi+\psi)} h \\
&= \lim \Phi_{(\varphi+\psi)1_{\mathcal{A}_n}} h \\
&= \lim \Phi_{\varphi+\psi} E_{\mathcal{A}_n} h \\
&= \lim \Phi_{\varphi+\psi} h_n = \lim(\Phi_\varphi h_n + \Phi_\psi h_n)
\end{aligned}$$

□

**Lemma 4.14.** *For $f \in L^0(\Phi)$, one has*

$$\Phi_f \in \mathscr{L}(\mathcal{H}) \text{ if and only if } f \in L^\infty(\Phi).$$

*Proof.* " $\Longleftarrow$ " already proven in Lemma 4.8.

For the other direction we prove that $\|\Phi_f\| \geq \|f\|_\infty$. Let $\lambda < \|f\|_\infty$. Then $\mathcal{A}_\lambda := \{|f| \geq \lambda\}$ is not a $\Phi$-null set. By the polarization identity, there exists a $h \in \mathcal{H}$, such that $\mathcal{A}_\lambda$ is not a $\mathfrak{m}_{h,h}$-null set. We have

$$\mathcal{A}_\lambda = \bigcup_{\substack{\mu \in \mathbb{Q} \\ \mu > \lambda}} \{\lambda \leq |f| \leq \mu\}$$

which gives

$$0 \neq \mathfrak{m}_{h,h}(\mathcal{A}_\lambda) = \sup_{\substack{\mu \in \mathbb{Q} \\ \mu > \lambda}} \mathfrak{m}_{h,h}(\{\lambda \leq |f| \leq \mu\}).$$

Therefore, there exists a $\mu > \lambda$, such that

$$\mathfrak{m}_{h,h}(\{\lambda \leq |f| \leq \mu\}) =: \mathfrak{m}_{h,h}(B) > 0.$$

We then have that $E_B h \in \mathfrak{D}(f)$, since $f$ is bounded on $B$. Note that this is a priori not true for $\mathcal{A}_\lambda$.

$$\begin{aligned}\|\Phi_f E_B h\|^2 &= \int |f|^2 \,\mathrm{d}\mathfrak{m}_{E_Bh,E_Bh} \\ &= \int_B |f|^2 \,\mathrm{d}\mathfrak{m}_{h,h} \\ &\geq \lambda^2 \mathfrak{m}_{h,h}(B) \\ &= \lambda^2 \int \overline{1}_B 1_B \,\mathrm{d}\mathfrak{m}_{h,h} \\ &= \lambda^2 \int \mathrm{d}\mathfrak{m}_{E_Bh,E_Bh} = \lambda^2 \|E_B h\|^2\end{aligned}$$

For $\tilde{h} := \frac{E_B h}{\|E_B h\|} \in \mathfrak{D}(f)$, we have $\|\Phi_f \tilde{h}\| \geq \lambda$. Since $\|\tilde{h}\| = 1$, $\|\Phi_f\| \geq \lambda$. □

**Lemma 4.15.** *For $f \in L^0(\Phi)$, it holds that $\Phi_f$ is invertible if and only if $\{f = 0\}$ is a $\Phi$-null set and $1/f \in L^\infty(\Phi)$. One then has $\Phi_f^{-1} = \Phi_{1/f}$*

*Proof.* $\Leftarrow$: By the previous Lemma $\Phi_{1/f} \in \mathscr{L}(\mathcal{H})$, and by Lemma 4.13

$$\Phi_{1/f}\Phi_f \subset \Phi_{1/f f} = \Phi_1 = \mathrm{id} = \Phi_{f 1/f} = \Phi_f \Phi_{1/f}.$$

Hence, $\Phi_f$ is invertible.

$\Rightarrow$: Let $h \in E_{\{f=0\}}(\mathcal{H})$. Since $E_{\{f=0\}}$ is a projection, $E_{\{f=0\}}h = h$. Thus

$$\Phi_f h = \Phi_f E_{\{f=0\}} h = \Phi_{f \cdot 1_{\{f=0\}}} h = \Phi_0 h = 0.$$

Since $\Phi_f$ is invertible, $h = 0$ and $E_{\{f=0\}} = 0$ are immediate consequences. Therefore $\{f = 0\}$ is a $\Phi$-null set.

It remains to show that $1/f \in \mathrm{L}^\infty(\Phi)$. We have

$$\Phi_f \cdot \Phi_{1/f} \subset \Phi_1 = I.$$

On $\mathfrak{D}(\Phi_f \cdot \Phi_{1/f})$, it holds that $\Phi_{1/f} = \Phi_f^{-1}$ . Since $\Phi_f$ is invertible, it is surjective. We claim $\mathcal{H} = \Phi_f(\mathfrak{D}(f)) \subset \mathfrak{D}(\Phi_{1/f})$

$$\begin{aligned}\int \left|\frac{1}{f}\right| \mathrm{d}|\mathfrak{m}_{g,\Phi_f h}| &= \sup_{|\phi| \leq |1/f|} \left| \int \phi \,\mathrm{d}\mathfrak{m}_{g,\Phi_f h} \right| \\ &= \sup_{|\phi| \leq |1/f|} \left| \int \phi f \,\mathrm{d}\mathfrak{m}_{g,h} \right| \leq \int \mathrm{d}|\mathfrak{m}_{g,h}| \leq \|h\|\|g\|.\end{aligned}$$

Thus $\mathfrak{D}(\Phi_{1/f}) = \mathcal{H}$, which implies $1/f \in \mathrm{L}^\infty(\Phi)$, by Lemma 4.14. □

**Lemma 4.16.** *Let $f \in L^0(\Phi)$. Then*

$$\operatorname{Sp} \Phi_f = \bigcap_{E_\mathcal{A} = I} \overline{f(\mathcal{A})}$$

*Proof.* " $\subset$ " Fix $\lambda \in \operatorname{Sp}\Phi_f$, $\mathcal{A} \subset X$ such that $E_\mathcal{A} = I$. We claim that $\lambda \in \overline{f(\mathcal{A})}$. By Lemmata 4.13 and 4.15, $\Phi_f - \lambda I = \Phi_{f-\lambda}$ is not invertible implies that either

*a)* $\{f - \lambda = 0\}$ is not a $\Phi$-null set.

*b)* $\{f - \lambda = 0\}$ is a $\Phi$-null set, but $1/(f-\lambda) \notin \mathrm{L}^\infty(\Phi)$.

Suppose $a$) holds. Since $\mathcal{A}^c$ is a $\Phi$-null set,

$$\mathcal{A} \cap \{f = \lambda\} \neq \varnothing.$$

This means there exists a $x \in \mathcal{A}$ such that $f(x) = \lambda$, that is

$$\lambda \in f(\mathcal{A}).$$

Suppose now that $b$) holds. By Lemma 4.15 ${}^1\!/\!{}_{(f-\lambda)} \notin \mathrm{L}^\infty(\Phi)$ implies that ${}^1\!/\!{}_{(f-\lambda)}$ is unbounded on $\mathcal{A} \setminus \{f = \lambda\}$. Thus, there exists a sequence $(x_n) \in \mathcal{A} \setminus \{f = \lambda\}$, such that

$$\lim |f(x_n) - \lambda| = 0,$$

which implies that

$$\lambda = \lim f(x_n) \in \overline{f(\mathcal{A})}.$$

"$\supset$" Fix $\lambda \notin \mathrm{Sp}(\Phi_f)$. We have to show that there exists a set $\mathcal{A}_0$, $E_{\mathcal{A}_0} = I$ and $\lambda \notin \overline{f(\mathcal{A}_0)}$. By Lemma 4.15, $\Phi_f - \lambda I = \Phi_{f-\lambda}$ is invertible implies that $\{f = \lambda\}$ is a $\Phi$-null set, and ${}^1\!/\!{}_{f-\lambda} \in \mathrm{L}^\infty(\Phi)$. Thus with $M := \|{}^1\!/\!{}_{f-\lambda}\|_\infty$, $\{|f - \lambda| < {}^1\!/\!{}_M\}$ is a $\Phi$-null set. Therefore, we can set $\mathcal{A}_0 := \{|f - \lambda| \geq {}^1\!/\!{}_M\}$. Then $E_{\mathcal{A}_0} = I$ and $d(\lambda, \overline{f(\mathcal{A}_0)}) \geq {}^1\!/\!{}_M$, that is

$$\lambda \notin \overline{f(\mathcal{A}_0)}.$$

□

Summing up the previous lemmata, we get our

**Theorem 4.17** (Main theorem)**.**

1. *If $f \in L^\infty(\Phi)$, then $\mathfrak{D}(f) = \mathcal{H}$, $\Phi_f \in \mathscr{L}(\mathcal{H})$ and*

$$\|\Phi_f\| = \|f\|_\infty = \inf \left\{ \alpha > 0 \;\middle|\; \alpha \geq |f|,\ \Phi\text{-}a.e. \right\}.$$

2. *Let $f \in L^0(\Phi), (f_n)$ a net in $L^\infty(\Phi)$, and $\alpha, \beta \geq 0$ such that*

$$|f_n| \leq \alpha|f| + \beta \textit{ for all } n$$
$$f_n \to f \;\; \Phi\text{-}a.e..$$

   *Then the following statements about $h \in \mathcal{H}$ are equivalent:*

   *(i)* $h \in \mathfrak{D}(f)$
   *(ii)* $\int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} < \infty$
   *(iii)* $(\Phi_{f_n} h)$ *converges in* $\mathcal{H}$

   *One then has*

$$\|\Phi_f h\|^2 = \int |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} \textit{ and}$$
$$\Phi_f h = \lim \Phi_{f_n} h.$$

3. *For each $f \in L^0(\Phi)$, $\Phi_f$ is a normal Operator, and we have*

$$\Phi_f^* = \Phi_{\bar{f}}.$$

   *If $f$ is real valued, then $\Phi_f$ is self-adjoint, and $E_A := \Phi_{1_A}$ for $A \in \mathscr{E}(X)$ is an orthogonal projection.*

4. *(i)* $f \in L^\infty(\Phi)$,

$$f \geq 0 \ \Phi\text{-}a.e. \Rightarrow \Phi_f \geq 0$$

*(ii)* $\mathcal{A} \in \mathscr{E}(X)$,

$$E_{\mathcal{A}} = 0 \Leftrightarrow \mathcal{A} \ \Phi\text{-}null\ set$$

*(iii)* $U \in \mathscr{E}(X)$ *open,*

$$U \neq \varnothing \Rightarrow E_U \neq 0$$

5. *For each* $f, g \in L^0(\Phi)$, $\alpha \in \mathbb{C}$, *we have*

   *(i)* $\Phi_{\alpha f} = \alpha \Phi_f$

   *(ii)* $\overline{\Phi_f + \Phi_g} = \Phi_{f+g}$

   *(iii)* $\mathfrak{D}(\Phi_f \Phi_g) = \mathfrak{D}(fg) \cap \mathfrak{D}(g)$, *and* $\overline{\Phi_f \Phi_g} = \Phi_{fg}$

   *(iv) if* $g \in L^\infty(\Phi)$ *then* $\Phi_f + \Phi_g = \Phi_{f+g}$, *and* $\Phi_f \Phi_g = \Phi_{fg}$.

6. *For* $f \in L^0(\Phi)$, *one has*

$$\Phi_f \in \mathscr{L}(\mathcal{H}) \text{ if, and only if, } f \in L^\infty(\Phi).$$

7. *For* $f \in L^0(\Phi)$, *we have*

$$\operatorname{Sp} \Phi_f = \bigcap_{A \in \mathscr{E}(X)} \overline{f(A)};$$

*where,* $A$ *runs over all* $A \in \mathscr{E}(X)$ *such that* $E_A = I$ *and* $A \subset \mathfrak{D}(f)$.

# 5 Spectral Theorem for Unbounded Operators

Using the results from Chapter 4, we construct a spectral measure for unbounded normal operators, completing the process started in chapter 3. This section follows [3].

**Theorem 5.1** (Spectral Theorem)**.** *Let $T$ be a normal operator on $\mathcal{H}$.*

$$\overline{\operatorname{Sp} T}^{\mathbb{C}} = \operatorname{Sp} T \cup \{\infty\} \Leftrightarrow T \text{ is unbounded.}$$

*There exists a uniquely determined spectral measure*

$$\Phi : \mathrm{C}\left(\overline{\operatorname{Sp} T}^{\mathbb{C}}\right) \to \mathscr{L}(\mathcal{H})$$

*such that*

*(i)* $\{\infty\}$ *is a* $\Phi$*-zeroset*

*(ii)* $\Phi_{\mathrm{id}} = T$*, where* $\mathrm{id}(\infty) := 0$*, which is arbitrary.*

*Proof.* In Proposition 3.7, we defined the inverse Gelfandisomorphism

$$\Phi : \mathrm{C}(\theta(\operatorname{Sp} \mathfrak{A})) \to \mathfrak{A} \subset \mathscr{L}(\mathcal{H}),\ g \mapsto \mathcal{G}^{-1}(g \circ \theta),$$

where $\mathfrak{A}(T) := \langle I, A, B, B^* \rangle\,,\ A := (I + T^*T)^{-1},\ B := TA$.

It followed that $\Phi$ is a spectral measure such that

$$A = \Phi_a,\ a : \lambda \mapsto \frac{1}{1 + |\lambda|^2},$$
$$B = \Phi_b,\ b : \lambda \mapsto \frac{\lambda}{1 + |\lambda|^2}.$$

We need to show that $\{\infty\}$ is a $\Phi$-zeroset. To do that, it suffices to show that $E_{\{\infty\}} = 0$. We have

$$AE_{\{\infty\}} = \Phi_a \Phi_{1_{\{\infty\}}} = \Phi_{a 1_{\{\infty\}}} = \Phi_0 = 0.$$

Because $A$ is the inverse of $1 + T^*T$, we get

$$E_{\{\infty\}} = (1 + T^*T) A E_{\{\infty\}} = 0.$$

Define $\mathrm{id} : \theta(\operatorname{Sp} \mathfrak{A}) \to \mathbb{C}$, via $\mathrm{id}(\infty) = 0$. Thus,

$$\mathrm{id} \in \mathrm{L}^0(\Phi), \text{ and } (1 + |\,\mathrm{id}\,|^2)a = 1 \quad \Phi\text{-a.e.}$$

Using the fact that $a$ is bounded and Lemma 4.14 , we get

$$I = \Phi_1 = \Phi_{(1+|\,\mathrm{id}\,|^2)a} = \Phi_{(1+|\,\mathrm{id}\,|^2)} \Phi_a.$$

Furthermore

$$\begin{aligned} I + TT^* =& \Phi_{(1+|\,\mathrm{id}\,|^2)} \Phi_a (I + TT^*) \\ =& \Phi_{(1+|\,\mathrm{id}\,|^2)} A (I + TT^*) \\ \subset& \Phi_{(1+|\,\mathrm{id}\,|^2)}, \end{aligned}$$

and

$$T = (I + T^*T)AT \subset \Phi_{(1+|\operatorname{id}|^2)} TA = \Phi_{(1+|\operatorname{id}|^2)} B.$$

Since $T$ is normal and hence closed, and $\Phi_{(1+|\operatorname{id}|^2)\cdot b} = \Phi_{\operatorname{id}}$, we get that

$$T = \Phi_{\operatorname{id}}.$$

It still remains to show that

$$\theta(\operatorname{Sp}\mathfrak{A}) = \overline{\operatorname{Sp} T}^{\mathbb{C}}.$$

By Lemma 4.16 , we have

$$\operatorname{Sp} T = \operatorname{Sp}(\Phi_{\operatorname{id}}) = \bigcap_{E_U = I} \overline{\operatorname{id}(U)}^{\mathbb{C}}.$$

Let $U \subset \theta\operatorname{Sp}(\mathfrak{A}) \subset \overline{\mathbb{C}}$, such that $U^{\mathsf{c}}$ is a $\Phi$-zeroset. Then

$$\begin{aligned} \operatorname{id}(U) &= U && \text{if } \infty \notin U \\ &= (U \cup \{0\}) \setminus \{\infty\} && \text{if } \infty \in U, \end{aligned}$$

giving us

$$\begin{aligned} \overline{\operatorname{id}(U)}^{\mathbb{C}} &= \overline{U}^{\mathbb{C}} && \text{if } \infty \notin U \\ &= (\overline{U}^{\mathbb{C}} \cup \{0\}) \setminus \{\infty\} && \text{if } \infty \in U. \end{aligned}$$

By Lemma 4.13 , $U^{\mathsf{c}}$ does not contain any open sets. Thus

$$\overline{U}^{\mathbb{C}} = \theta(\operatorname{Sp}\mathfrak{A}) \setminus \{\infty\},$$

and for each $U$, such that $E_U = I$

$$\theta(\operatorname{Sp}\mathfrak{A}) \setminus \{\infty\} \subset \overline{\operatorname{id}(U)}^{\mathbb{C}} \subset (\theta(\operatorname{Sp}\mathfrak{A}) \cup \{0\}) \setminus \{\infty\}.$$

For $U_0 := \theta(\operatorname{Sp}\mathfrak{A}) \setminus \{\infty\}$, it holds that $E_{U_0} = I$ and $\operatorname{id}(U_0) = U_0$, and therefore It follows that

$$\operatorname{Sp} T = \theta(\operatorname{Sp}\mathfrak{A}) \setminus \{\infty\}.$$

If $T$ is bounded, $\operatorname{Sp} T$ is bounded as well, and hence compact, which gives

$$\overline{\operatorname{Sp} T} = \operatorname{Sp} T.$$

Since $T$ is bounded, $I + T^*T = A^{-1}$ is bounded as well, and since $I + T^*T$ is invertible in $\mathscr{L}(\mathcal{H})$ it is invertible in $\mathfrak{A}$ by Proposition 2.1. Thus $I + T^*T$ lies in $\mathfrak{A}$. $I = (I + T^*T)A$ now implies

$$1 = \chi_\infty(I) = \chi_\infty(I + T^*T)\chi_\infty(A) = \chi_\infty(I + T^*T) \cdot 0 = 0,$$

a contradiction. Thus $\operatorname{Sp} T = \theta(\operatorname{Sp}\mathfrak{A})$, if $T$ is bounded.

If on the other hand, $\operatorname{Sp} T$ is compact in $\mathbb{C}$, then

$$\operatorname{id} \in \mathrm{L}^\infty(\Phi), \text{ and so } T = \Phi_{\operatorname{id}} \in \mathcal{B}(\mathcal{H}).$$

Thus, we have proven

$$T \text{ is bounded } \Leftrightarrow \operatorname{Sp} T \text{ compact in } \mathbb{C}.$$

If now $T$ is unbounded, then $\operatorname{Sp} T$ is not compact in $\mathbb{C}$, and hence

$$\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}} = \operatorname{Sp} T \cup \{\infty\}\,.$$

Since $\theta(\operatorname{Sp}\mathfrak{A}) \subset \overline{\mathbb{C}}$ is compact, we get

$$\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}} = \theta(\operatorname{Sp}\mathfrak{A}).$$

Thus there exists a spectral measure

$$\Phi : \mathrm{C}(\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}}) \to \mathscr{L}(\mathcal{H}).$$

To check uniqueness, let $\Phi'$ be another spectral measure, such that $\infty$ is a $\Phi'$-zeroset, and $\Phi'(\mathrm{id}) = T$. To prove $\Phi = \Phi'$, by Stone-Weierstraß we only need to show

$$\Phi'_a = \Phi_a = A,\ \Phi'_b = \Phi_b = B.$$

We know

$$T^*T = (\Phi'_{\mathrm{id}})^*\Phi'_{\mathrm{id}} \subset \Phi_{|\,\mathrm{id}\,|^2}.$$

Since $T^*T$ is normal and hence closed, we have equality.

$$\begin{aligned}
\Phi'_a(I + T^*T) = \Phi'_a\Phi'_{1+|\,\mathrm{id}\,|^2} \subset \Phi'_{a(1+|\,\mathrm{id}\,|^2)} &= I \\
= \Phi'_{a(1+|\,\mathrm{id}\,|^2)} = \Phi'_a\Phi'_{(1+|\,\mathrm{id}\,|^2)} &= (I - T^*T)\Phi'_a,
\end{aligned}$$

that is

$$\begin{aligned}
\Phi'_a &= (I + T^*T)^{-1} = A \\
\Phi'_b &= \Phi'_{\mathrm{id}\,\cdot a} = \Phi'_{\mathrm{id}}A = TA = B.
\end{aligned}$$

□

# 6 Applications

Our applications are motivated by quantum physics. Two fundamental operators in quantum mechanics are the momentum operator $P \coloneqq i\frac{\partial}{\partial x}$ , and the position operator $M_x(f)(x) = xf(x)$.

**Remark 6.1.** The study of these operators relies heavily on the chosen Hilbert space $\mathcal{H}$, as $\operatorname{Sp} M_x$ depends on it.

**Example 6.2.** Let $\mathcal{H} \coloneqq \mathrm{L}^2([0,1])$. $M_x \in \mathcal{B}(\mathcal{H})$ by Hölder's inequality: Let $f \in \mathcal{H}$

$$\|M_x(f)\|^2 = \int_0^1 x^2 |f(x)|^2 \,\mathrm{d}\,x \leq 1 \cdot \int_0^1 |f(x)|^2 \,\mathrm{d}\,x = \|f\|^2.$$

Thus $\|M_x\| \leq 1$. Furthermore $M_x$ is self-adjoint, which implies that $\operatorname{Sp} M_x \subset [-1,1]$. But for $\lambda > 0$, $M_{x-\lambda}$ is invertible as $M_{1/(x-\lambda)} \in \mathcal{B}(\mathcal{H})$, again by Hölder. Therefore $\operatorname{Sp} M_x \subset [0,1]$. To show $\operatorname{Sp} M_x \supset [0,1]$, let $\lambda \in [0,1]$. Again the inverse of $M_{x+\lambda}$ would be $M_{1/(x+\lambda)}$, the latter not being bounded, as $\frac{1}{x+\lambda} \notin \mathrm{L}^2([0,1])$. We conclude that $x$ does not a have a preimage in $\mathcal{H}$ under $M_{x+\lambda}$. Hence $\operatorname{Sp} M_x = [0,1]$.

Our Spectral Theorem 2.3 states, that the map $\Phi : \mathrm{C}([0,1]) \to \mathrm{Sp}(\langle M_x, I\rangle)$, is an isometry. This isometry sends id to $M_x$, and the constant function 1 to $I$. We conclude that $\phi$ is mapped via $\Phi$ to $M_{\phi(x)} : g \mapsto \phi \cdot g$. In this example, it is obvious that, $\mathrm{L}^\infty$ is mapped to $\mathcal{B}(\mathcal{H})$. If we take $\phi(x) = 1/x$, $\mathfrak{D}(\phi)$ is by definition, all elements $f \in \mathcal{H}$, such that $g \mapsto \langle g, M_\phi f\rangle$ is continuous in $g$ for all $g \in \mathcal{H}$. Since we have such an explicit form of the operator, one readily sees via Lemma 4.10 (*ii*) that $\mathfrak{D}(\phi) = \{f \in \mathcal{H} \mid \|M_\phi(f)\|^2 < \infty\}$. As seen in Example 2.5, or by direct calculation, $M_\phi$ is the inverse to $M_x$. $M_\phi$ is unbounded, and $M_x$ is the bounded inverse to $M_x$. This is an example, where we started with a bounded operator, and via the Functional Calculus for measurable functions, got an unbounded operator

If we change the Hilbert space to $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$, the spectrum of our multiplication operator changes to $\mathrm{Sp}(M_x) = \mathbb{R}$. Thus, $M_x$ is unbounded. Its inverse is again $M_{1/x}$. Note that both operators are unbounded, and hence, neither is boundedly invertible.

**Example 6.3.** Let $\mathcal{H} \coloneqq \mathrm{L}^2([0,1])$. The operator we want to consider is the momentum operator $P = i\frac{\partial}{\partial x}$. We recall the definition of a normal operator. $P$ is called normal if $\mathfrak{D}(P) = \mathfrak{D}(P^*)$ and $\|P\| = \|P^*\|$. To get a normal operator, we need to specify boundary conditions on $\mathfrak{D}(P)$. One knows that $P^*$ acts the same way as $P$. Therefore, if $P$ is normal, it is self-adjoint. We compute

$$\begin{aligned}
\langle f, Pg\rangle &= \int_0^1 \overline{f(x)} i \frac{\partial}{\partial x} g(x) \mathrm{d}x = \overline{f(x)} i g(x)\Big|_0^1 - \int_0^1 i \frac{\partial}{\partial x}\overline{f(x)} g(x) \mathrm{d}x \\
&= i(\overline{f(1)}g(1) - \overline{f(0)}g(0)) + \int_0^1 \frac{\partial}{\partial x}\overline{if(x)} g(x) \mathrm{d}x \\
&= i(\overline{f(1)}g(1) - \overline{f(0)}g(0)) + \langle Pf, g\rangle\,.
\end{aligned}$$

For $P$ to be self-adjoint, the left term from the last line needs to vanish. Thus, we get the boundary conditions $\{f \in H^1([0,1]) \mid f(0) = f(1)\}$. To determine the spectrum of the operator, we look at the eigenvectors of $(P+\lambda I)$. $(P+\lambda I)(f) =$

$f$ reformulates to $if' = i(\lambda - 1)f$, giving $f(x) = \mathrm{e}^{i(\lambda-1)x}$. This is in the domain of $P$, if $\lambda = 2\pi k$ for $k \in \mathbb{Z}$. We claim that these elements already form a Hilbert basis. To see this, we periodically extend any $f \in \mathcal{H} := \mathrm{L}^2([0,1])$ to $\mathrm{L}^2(\mathbb{R})$. Now we can consider functions on the quotient $\mathbb{R}\backslash\mathbb{Z} = \mathbb{S}^1$. For $\mathrm{L}^2(\mathbb{S}^1)$, $\left(\mathrm{e}^{2\pi ikx}\right)_{k\in\mathbb{Z}}$ forms a Hilbert basis, via Fourier expansion. A full proof can be found in [8, Ch. V.4]. Using the Fourier transform $\mathcal{F}$, $P$ becomes the multiplication operator $M_{2\pi k}$: For $f \in \mathfrak{D}(P)$, we get

$$\begin{aligned}(\mathcal{F}(Pf))(k) &= \frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{2\pi ikx}\, i(\partial_x f)(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^1 2\pi k\, \mathrm{e}^{2\pi ikx}\, f(x)\mathrm{d}x \\ &= 2\pi k(\mathcal{F}f)(k).\end{aligned}$$

The Fourier transform maps $\mathrm{L}^2(\mathbb{S}^1)$ isometrically to $\ell^2(\mathbb{Z})$. $\mathrm{Sp}(M_{2\pi k}) = 2\pi\mathbb{Z}$, as $(M_{2\pi k} + \lambda I)^{-1} = (M_{2\pi k+\lambda})^{-1} = M_{(2\pi k+\lambda)^{-1}}$, for $\lambda \notin 2\pi\mathbb{Z}$.

The map $\Phi : \mathrm{Sp}(P) \to \mathscr{L}(\mathcal{H}),\ f \mapsto \mathcal{F}M_{f(2\pi k)}\mathcal{F}^{-1}$ satisfies the conditions of being a spectal measure for the operator $P$. By Theorem 5.1 it is the unique spectral measure for $P$. We can therefore apply the arguments from Example 6.2, to $M_{2\pi k}$, and transform back.

The fact that we could find another Hilbert space, such that our operator acts as a multiplication operator, was not mere chance. A more general version of the spectral theorem states the following:

**Theorem 6.4** (Spectral Theorem, multiplication operator)**.** *Let $T$ be a normal operator on a Hilbert space $\mathcal{H}$. There exists a measure space $(\Omega, \Sigma, \mu)$, a measurable function $f : \Omega \to \mathbb{C}$, and a unitary operator $U : \mathcal{H} \to L^2(\mu)$, such that*

(a) *$x \in \mathfrak{D}(T)$ if, and only if $f \cdot U(x) \in L^2(\mu)$.*

(b) *$UTU^*\phi = f \cdot \phi = M_f(\phi)$ for $\phi \in \mathfrak{D}(M_f) = \left\{\phi \in L^2(\mu) \mid f\phi \in L^2(\mu)\right\}$.*

The proof for self-adjoint $T$ can be found in [8, Ch. VII.4]. Note, that if $T$ is self-adjoint, then the function $f$ from the theorem, takes only real values.

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