

# COUNTING COVERS OF ELLIPTIC CURVES

Orlando Marigliano

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# 1. Quasimodular forms

This section introduces quasimodular forms as described in [KZ].

## 1.1. The space of modular forms

Let  $\mathcal{H} = \{\tau \in \mathbb{C}; \operatorname{Im}(\tau) > 0\}$  denote the upper half-plane. For  $\tau \in \mathcal{H}$ , define  $q = \exp(2\pi i\tau)$  and  $Y = 4\pi \operatorname{Im}(\tau)$ . Further, let  $\operatorname{SL}_2(\mathbb{Z}) \subset \operatorname{SL}_2(\mathbb{C})$  denote the full modular group. Then  $\operatorname{SL}_2(\mathbb{Z})$  acts on  $\mathcal{H}$  by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}).$$

**Definition 1.1.** Let  $f: \mathcal{H} \rightarrow \mathbb{C}$  be a function, let  $k \in \mathbb{Z}$ .

1. The function  $f$  is  $\mathbb{Z}$ -periodic, if it satisfies  $f(\tau + 1) = f(\tau)$  for all  $\tau \in \mathcal{H}$ . In this case there exists a function  $\tilde{f}: B \setminus \{0\} \rightarrow \mathbb{C}$ , defined on the open unit ball  $B \subset \mathbb{C}$  with the origin removed, such that  $f(\tau) = \tilde{f}(q)$  for all  $\tau$ . Now let  $f$  be holomorphic. Then so is  $\tilde{f}$ . We say that  $f$  is *holomorphic at infinity*, if  $\tilde{f}$  has a holomorphic continuation to the whole of  $B$ .
2. The function  $f$  is said to satisfy the *modular condition of weight  $k$* , if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

for all  $\tau$  in  $\mathcal{H}$  and all  $\gamma \in \operatorname{SL}_2(\mathbb{Z})$ . Such a function is  $\mathbb{Z}$ -periodic, as can be seen by setting  $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

3. The function  $f$  is a *modular form (of weight  $k$ )* if it is holomorphic, satisfies the modular condition and is holomorphic at infinity.

Note that if  $k$  is odd, then any function satisfying the modular condition of weight  $k$  is zero. This follows by using the modular condition with  $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ . There are several alternate conventions for handling the weights  $k$ . Some authors for instance replace  $k$  by  $2k$  throughout, so that “modular forms of weight  $2k$ ” are considered. This is the convention used by [Sera].

The modular forms of weight  $k$  form a vector space, denoted<sup>1</sup> by  $M_k$ . Multiplying two modular forms of weights  $k$  respectively  $l$  yields a modular form of weight  $k + l$ , giving the space  $\bigoplus_k M_k$  the structure of a graded ring, denoted by  $M_*$ .

**Examples 1.2.** For an even integer  $k \geq 2$ , the *Eisenstein series*<sup>2</sup> of weight  $k$  is the function

$$E_k(\tau) = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

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<sup>1</sup> In [Sera], the space of modular forms of weight  $2k$  is denoted by  $M_k$ .

where  $b_k$  is the  $k$ -th Bernoulli number, and  $\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$ . By definition, these functions are holomorphic at infinity.

For  $k \geq 4$ , the Eisenstein series of weight  $k$  is a modular form of weight  $k$ . One proves this for example by showing that for  $k \geq 4$ , the series  $E_k$  is a multiple of the function  $G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} (m\tau + n)^{-k}$ , which is indeed modular of weight  $k$ , see [Sera, Ch. VII, Prop. 8] and [Sera, Ch. VII, 2.3].

The function  $\Delta = 2^{-6}3^{-3}(E_4^3 - E_6^2)$  is a modular form of weight 12. By a theorem of Jacobi [Sera, Ch. VII, Thm. 6], one has

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

**Proposition 1.3.** *There is an isomorphism of graded rings*

$$\mathbb{C}[X_4, X_6] \xrightarrow{\sim} M_*$$

mapping  $X_i$  to  $E_i$ , where the former ring is graded by assigning to  $X_i$  the degree  $i$ . In particular, there are no nonzero modular forms of negative weight.

*Proof.* See [Sera, Ch. VII, 3.1, 3.2] □

## 1.2. The space of quasimodular forms

Let  $\mathcal{O}(\mathcal{H})$  denote the vector space of  $\mathbb{C}$ -valued holomorphic functions on  $\mathcal{H}$ . Recall the imaginary part function  $Y(\tau) = 4\pi \operatorname{Im}(\tau)$ . The following proposition shows that one may compare coefficients of elements of  $\mathcal{O}(\mathcal{H})[Y^{-1}]$  as if  $Y$  was a formal variable.

**Proposition 1.4.** *Let  $F = \sum_{m=0}^M f_m Y^{-m}$  be an element of  $\mathcal{O}(\mathcal{H})[Y^{-1}]$ . If  $F = 0$ , then  $f_m = 0$  for all  $m$ .*

*Proof.* For the differential operator  $\frac{d}{d\bar{\tau}}$  one has  $\frac{d}{d\bar{\tau}} Y^{-m} = -2\pi i m Y^{-m-1}$  and  $\frac{d}{d\bar{\tau}} f_m = 0$ , hence

$$0 = \frac{d}{d\bar{\tau}} F(\tau) = -2\pi i \sum_{m=1}^M f_m(\tau) Y^{-m-1} = -2\pi i Y^{-2} \left( \sum_{m=0}^{M-1} f_{m+1} \tau Y^{-m} \right).$$

By induction this implies that the  $f_m$  are zero for  $m \geq 1$ , hence also  $f_0 = 0$ . □

**Corollary 1.5.** *Let  $F = \sum_{m=0}^M f_m Y^{-m}$  be an element of  $\mathcal{O}(\mathcal{H})[Y^{-1}]$  satisfying the modular condition of weight  $k$ . Then the  $f_m$  are  $\mathbb{Z}$ -periodic.*

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<sup>2</sup>In [Sera], the Eisenstein series of weight  $k$  as defined below is denoted by  $E_{k/2}$ . A similar remark applies to the function  $G_k$  below.

**Definition 1.6.** An *almost holomorphic modular form (of weight  $k$ )* is an element

$$F = \sum_{m=0}^M f_m Y^{-m}$$

of  $\mathcal{O}(\mathcal{H})[Y^{-1}]$  such that  $F$  satisfies the modular condition and the  $f_m: \mathcal{H} \rightarrow \mathbb{C}$  are holomorphic at infinity.

**Proposition 1.7.** *Let  $F(\tau) = \sum_{m=0}^M f_m(\tau)Y^{-m}$  be an almost holomorphic modular form. Then the leading coefficient  $f_M$  is a modular form of weight  $k - 2M$ . In particular, if  $f_M \neq 0$ , then  $2M \leq k$ .*

*Proof.* This follows after comparing the coefficients of  $Y^{-M}$  in both sides of the modularity condition  $F(\gamma\tau) = (c\tau + d)^k F(\tau)$ , using the equality

$$Y^{-1}(\gamma\tau) = (c\tau + d)^2 Y(\tau)^{-1} + \frac{c(c\tau + d)}{2\pi i}$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$ . □

The almost holomorphic modular forms of weight  $k$  form a vector space, denoted by  $\widehat{M}_k$ . Let  $\widehat{M}_*$  denote the associated graded ring.

**Definition 1.8.** An element in the image of the map  $\widehat{M}_k \rightarrow \mathcal{O}(\mathcal{H})$  taking an almost holomorphic modular form  $F = \sum_{m=0}^M f_m Y^{-m}$  of weight  $k$  to  $f_0$  is called a *quasimodular form of weight  $k$* . Hence a quasimodular form is a holomorphic function on the upper plane appearing as the constant term of an almost holomorphic modular form.

Again, denote the vector space of quasimodular forms of weight  $k$  by  $\widetilde{M}_k$  and the associated graded ring by  $\widetilde{M}_*$ . The definition gives a surjective graded ring homomorphism  $\widehat{M}_* \rightarrow \widetilde{M}_*$  and one has  $\widehat{M}_k \cap \widetilde{M}_k = M_k$ .

**Example 1.9.** Consider the second Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where  $\sigma_1(n) = \sum_{d|n} d$ . For the weight 12 modular form  $\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ , one has the identity  $2\pi i E_2(\tau) = \frac{d}{d\tau} \log(\Delta(\tau))$ , which is proven by a straightforward computation. Using the modularity of  $\Delta$ , one then computes

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c(c\tau + d)}{\pi i},$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$ .

Now, since  $Y^{-1}(\gamma\tau) = (c\tau + d)^2 Y(\tau)^{-1} + \frac{c(c\tau + d)}{2\pi i}$ , it follows that  $E_2^* = E_2 - 12/Y$  is an almost holomorphic modular form of weight 2. Hence,  $E_2$  is a quasimodular form of weight 2.

**Proposition 1.10.** *The space  $\widetilde{M}_*$  of quasimodular forms satisfies the following properties.*

1. *The canonical graded homomorphism  $\widehat{M}_* \rightarrow \widetilde{M}_*$  is an isomorphism.*
2. *There is an isomorphism of graded rings  $M_* \otimes \mathbb{C}[X_2] \simeq \mathbb{C}[X_2, X_4, X_6] \rightarrow \widetilde{M}_*$  mapping  $X_i$  to  $E_i$ , where the former ring is graded by assigning to  $X_i$  the degree  $i$ .*
3. *Quasimodular forms are closed under taking derivatives.*

*Proof.* 1. The map  $\widehat{M}_* \rightarrow \widetilde{M}_*$  is surjective by definition. Injectivity follows from Calculation 10.1. Given an almost holomorphic modular form  $F(\tau) = \sum_{m=1}^M f_m(\tau)Y^{-m}$  with constant term zero, the strategy is to solve the modularity equation for the coefficients  $f_m$ . This way, one finds for a fixed argument  $\tau$  a polynomial equation in the lower row components  $c, d$  of any transformation  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , involving the coefficients  $f_m(\tau)$ . By varying the transformation  $\gamma$ , one may force these coefficients to be zero.

2. Express the map  $\mathbb{C}[X_2, X_4, X_6] \rightarrow \widetilde{M}_*$  as the composition

$$\mathbb{C}[X_2^*, X_4, X_6] \rightarrow \widehat{M}_* \rightarrow \widetilde{M}_*,$$

where the first map takes  $X_2^*$  to  $E_2^*$  and  $X_i$  to  $E_i$ , and the second map is the canonical map, which is an isomorphism by the first point above.

To prove the surjectivity of the first map, let  $F(\tau) = \sum_{m=0}^M f_m(\tau)Y^{-m}$  be an almost holomorphic modular form. Then  $f_M(E_2^*/12)^M$  is an almost holomorphic modular form of weight  $k$ , since  $f_M$  is modular of weight  $k - 2M$ , and the difference  $F - f_M(E_2^*/12)^M$  has degree smaller than  $M$ . Now use induction on  $M$ .

To get injectivity, let  $F = \sum_{\alpha=0}^{k/2} (E_2^*)^\alpha f_{k-2\alpha}$  be an almost holomorphic modular form of weight  $k$ , in the image of the first map, where the  $f_m$  are modular of weight  $m$ . If  $F = 0$ , then by comparing the coefficients of  $Y^{-k/2}$  one obtains  $0 = f_0$ . Now it follows by induction on  $k$  that the other coefficients  $f_m$  are zero. Hence  $F$  was the image of the zero element in  $M_* \otimes \mathbb{C}[X_2^*]$ .

3. To prove the last statement, one verifies that  $(6/\pi i)E_2' - E_2^2$  is modular of weight 4, and that if  $f$  is modular of weight  $k$ , then  $(6/\pi i)f' - kE_2 f$  is modular of weight  $2 + k$ . Now use the second point above.

□

## 2. Modular curves

One may weaken the definition of a modular form by requiring that modular condition be met only for transformations lying in certain subgroups  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$ . In this section we will calculate the dimension of the associated space of weight  $k$  modular forms  $M_k(\Gamma)$  using the fact that modular forms can be seen as section of a certain line bundle on a special Riemann surface, the modular curve associated to the subgroup  $\Gamma$ . We will roughly follow [DS06, Ch. 1-3].

### 2.1. Congruence subgroups and modular forms

**Definition 2.1.** Let  $N \in \mathbb{Z}$ .

1. The *principal congruence subgroup* of level  $N$  is the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

2. A *congruence subgroup* is a subgroup  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$  such that  $\Gamma(N) \subset \Gamma$  for some  $N \in \mathbb{Z}$ . We then say that  $\Gamma$  has *level*  $N$ .

**Remark 2.2.** The subgroup  $\Gamma(N)$  is the kernel of the component-wise congruence map  $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ . It is hence normal in  $\mathrm{SL}_2(\mathbb{Z})$  and of finite index. Consequently, each congruence subgroup has finite index in  $\mathrm{SL}_2(\mathbb{Z})$ , while not being necessarily normal.

**Definition 2.3.** Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be an element of  $\mathrm{GL}_2(\mathbb{C})$  and let  $f: \mathcal{H} \rightarrow \widehat{\mathbb{C}}$  be a holomorphic function. For  $\tau \in \mathcal{H}$  define the *factor of automorphy*

$$j(\gamma, \tau) := c\tau + d$$

and for  $k \in \mathbb{Z}$  the function  $f[\gamma]_k: \mathcal{H} \rightarrow \mathbb{C}$  by

$$f[\gamma]_k(\tau) := \det(\gamma)^{k/2} j(\gamma, \tau)^{-k} f(\gamma\tau).$$

**Remark 2.4.** Let  $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$  and  $\tau \in \mathcal{H}$ .

1. The factor of automorphy satisfies  $j(\gamma\gamma', \tau) = j(\gamma, \gamma'(\tau))j(\gamma', \tau)$ .

2. For all holomorphic functions  $f: \mathcal{H} \rightarrow \mathbb{C}$ , we have  $f[\gamma\gamma']_k = (f[\gamma]_k)[\gamma']_k$ .

**Definition 2.5.** Let  $f: \mathcal{H} \rightarrow \mathbb{C}$  be a function, let  $\Gamma$  be a congruence subgroup. Hence  $\Gamma$  contains an element of the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma$ . Let  $h \in \mathbb{N}$  be minimal with the property that  $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ . For  $\tau \in \mathcal{H}$ , set  $q_h = \exp(2\pi i\tau/h)$ .

1. The function  $f$  is  *$h\mathbb{Z}$ -periodic*, if it satisfies  $f(\tau + h) = f(\tau)$  for all  $\tau \in \mathcal{H}$ . Analogously to the case  $h = 1$ , there exists a function  $\tilde{f}: B \setminus \{0\} \rightarrow \mathbb{C}$  such that

$f(\tau) = \tilde{f}(q_h)$  for all  $\tau$ . Now let  $f$  be holomorphic, so that  $\tilde{f}$  is also holomorphic. We say that  $f$  is *holomorphic at infinity*, if  $\tilde{f}$  has a holomorphic continuation to the whole of  $B$ .

2. The function  $f$  is said to satisfy the *modular condition of weight  $k$  with respect to  $\Gamma$* , if  $f[\gamma]_k = f$  for all  $\gamma \in \Gamma$ . Such a function is  $h\mathbb{Z}$ -periodic, since  $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ .

3. For all  $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ , the group  $\alpha^{-1}\Gamma\alpha$  is a conjugation subgroup. Now assume that the function  $f$  is holomorphic and that it satisfies the modular condition of weight  $k$  with respect to  $\Gamma$ . Hence, for all  $\alpha$ , the function  $f[\alpha]_k$  satisfies the modular condition of the same weight with respect to  $\alpha^{-1}\Gamma\alpha$ , and is hence  $h_\alpha\mathbb{Z}$ -periodic for some  $h_\alpha$ . We define  $f$  to be *holomorphic at all cusps of  $\Gamma$* , if for all  $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ , the function  $f[\alpha]_k$  is holomorphic at  $\infty$ .

4. The function  $f$  is a *modular form (of weight  $k$ ) with respect to  $\Gamma$*  if it is holomorphic, satisfies the modular condition and is holomorphic at all cusps of  $\Gamma$ .

5. The function  $f$  is a *cuspidal form of weight  $k$  with respect to  $\Gamma$* , if it is a modular form of weight  $k$ , and if the associated holomorphic function  $\tilde{f}$  satisfies  $\tilde{f}(0) = 0$  after its holomorphic continuation.

## 2.2. The topology of modular curves

Let  $\Gamma$  be a congruence subgroup.

### Definition 2.6.

1. For  $M > 0$ , define the set  $\mathcal{N}_M \subset \mathbb{C} \cup \{\infty\}$  by

$$\mathcal{N}_M = \{\tau \in \mathcal{H} ; \mathrm{Im}(\tau) > M\} \cup \{\infty\}.$$

2. Define the *compact upper half-plane  $\mathcal{H}^*$*  to be the set

$$\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\},$$

endowed with the topology generated by union of the topology of  $\mathcal{H}$  with the set

$$\{\alpha(\mathcal{N}_M) ; \alpha \in \mathrm{SL}_2(\mathbb{Q}), M \in \mathbb{R}_{>0}\}.$$

With this definition, the group  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $\mathcal{H}^*$  by continuous maps. This group action is transitive on the subset  $\mathbb{Q} \cup \{\infty\}$ .

3. The *modular curve  $X(\Gamma)$*  is defined as the quotient space  ${}_\Gamma\backslash\mathcal{H}^*$ . Denote the canonical projection map  $\mathcal{H}^* \rightarrow X(\Gamma)$  by  $\pi$ .

For  $\tau \in \mathcal{H}^*$ , denote the stabilizer of  $\tau$  under the action of  $\Gamma$  by  $\Gamma_\tau$ .



**Remark 2.7.**

1. Define the function  $s: \mathcal{H} \rightarrow \mathrm{SL}_2(\mathbb{R})$  by  $s(x + iy) = \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ . If  $\tau \in \mathcal{H}$ , then  $s(\tau)(i) = \tau$ .
2. The stabilizer of  $i$  with respect to the transitive group action of the topological group  $\mathrm{SL}_2(\mathbb{R})$  on  $\mathcal{H}$  is the compact subgroup  $\mathrm{SO}_2(\mathbb{R})$ .
3. Let  $e_1, e_2 \in \mathcal{H}$ . We have  $\gamma(e_1) = e_2$  if and only if  $\gamma \in s(e_2) \mathrm{SO}_2(\mathbb{R}) s(e_1)^{-1}$ .

**Proposition 2.8.** *If  $\tau_1, \tau_2 \in \mathcal{H}$ , then there are open neighborhoods  $U'_1$  of  $\tau_1$  and  $U'_2$  of  $\tau_2$ , such that for all but finitely many  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , the sets  $\gamma(U'_1)$  and  $U'_2$  do not meet.*

*Proof.* Choose  $U'_1$  and  $U'_2$  to be any open neighborhoods belonging to the topology of  $\mathcal{H}$ , and with compact closure. Let  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . The previous remark implies that  $\gamma(\overline{U'_1}) \cap \overline{U'_2} \neq \emptyset$  is equivalent to

$$\gamma \in \mathrm{SL}_2(\mathbb{Z}) \cap \bigcap_{\substack{e_1 \in \overline{U'_1} \\ e_2 \in \overline{U'_2}}} s(e_2) \mathrm{SO}_2(\mathbb{R}) s(e_1)^{-1}.$$

But this subgroup is compact and discrete, hence finite. □

**Proposition 2.9.** *Let  $\tau_1$  and  $\tau_2$  be elements of  $\mathcal{H}^*$ . Then there exist open neighborhoods  $U_1$  of  $\tau_1$  and  $U_2$  of  $\tau_2$  such that for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , if  $\gamma(U_1)$  meets  $U_2$  then  $\gamma\tau_1 = \tau_2$ .*

*Proof.* We consider three cases separately.

1. Let  $\tau_1, \tau_2 \in \mathcal{H}$ . Let  $U'_1$  and  $U'_2$  be open neighborhoods of  $\tau_1$  and  $\tau_2$ , respectively, such that the set

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}); \gamma(U'_1) \cap U'_2 \neq \emptyset, \gamma(\tau_1) \neq \tau_2\}$$

is finite. We denote this set by  $F$ . For each  $\gamma \in F$ , choose disjoint open neighborhoods  $U_{1,\gamma}$  and  $U_{2,\gamma}$  of  $\gamma\tau_1$  and  $\tau_2$ , respectively, and put

$$U_1 = U'_1 \cap \left( \bigcap_{\gamma \in F} \gamma^{-1}(U_{1,\gamma}) \right) \text{ and}$$

$$U_2 = U'_2 \cap \left( \bigcap_{\gamma \in F} U_{2,\gamma} \right).$$

Then  $U_1$  and  $U_2$  satisfy the required properties.

2. Let  $\tau_1 \in \mathbb{Q} \cup \{\infty\}$  and  $\tau_2 \in \mathcal{H}$ . Choose  $U_2$  to be any open neighborhood in  $\mathcal{H}$  with compact closure. Now, there is some  $M \geq 0$  such that  $\mathrm{SL}_2(\mathbb{Z})\overline{U_2} \cup \mathcal{N}_M = \emptyset$ .

Let  $\alpha \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\alpha(\infty) = \tau_1$ . Choose  $U_1 = \alpha(\mathcal{N}_M)$ . Then for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , the sets  $\gamma(U_2)$  and  $U_1$  are disjoint.

3. Let  $\tau_1, \tau_2 \in \mathbb{Q} \cup \{\infty\}$ . Let  $\alpha_1, \alpha_2 \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\alpha_1(\infty) = \tau_1$  and  $\alpha_2(\infty) = \tau_2$ . Choose  $U_1 = \alpha_1(\mathcal{N}_2)$  and  $U_2 = \alpha_2(\mathcal{N}_2)$ . Then  $U_1$  and  $U_2$  satisfy the required properties.  $\square$

**Corollary 2.10.** *Let  $\tau \in \mathcal{H}^*$ . Then there is an open neighborhood  $U$  of  $\tau$  such that for all  $\gamma \in \mathcal{H}$ , if  $\gamma(U)$  meets  $U$  then  $\gamma \in \Gamma_\tau$ .*

**Proposition 2.11.** *The modular curve  $X(\Gamma)$  is connected, compact, and Hausdorff.*

*Proof.* The connectedness of  $X(\Gamma)$  follows from the connectedness of  $\mathcal{H}^*$ . For compactness, define the subsets

$$\mathcal{D} = \{\tau \in \mathcal{H}; |\tau| \geq 1, \mathrm{Re}(\tau) \leq 1/2\}$$

and  $\mathcal{D}^* = \mathcal{D} \cup \{\infty\}$ . The subset  $\mathcal{D}^* \subset \mathcal{H}^*$  is compact and a fundamental domain for the  $\mathrm{SL}_2(\mathbb{Z})$ -action on  $\mathcal{H}^*$ . Since  $\Gamma$  has finite index in  $\mathrm{SL}_2(\mathbb{Z})$ , it follows that a possible fundamental domain for the  $\Gamma$ -action is given by the union of finitely many images of  $\mathcal{D}^*$  under elements of  $\mathrm{SL}_2(\mathbb{Z})$ . Therefore, the modular curve  $X(\Gamma)$  is compact. Finally, the Hausdorff property follows from the previous proposition.  $\square$

### 2.3. Modular curves as Riemann surfaces

The modular curve  $X(\Gamma)$  may be given the structure of a Riemann surface. The needed local data is summarized below. We use the following convention: a subgroup  $G$  of  $\mathrm{SL}_2(\mathbb{Z})$  need not contain the matrix  $-\mathrm{id}$ ; we denote the subgroup generated by  $G$  and  $\{-\mathrm{id}\}$  by  $\pm G$ .

**Proposition 2.12.**

1. For  $\tau \in \mathcal{H}$ , the isotropy group  $\Gamma_\tau$  is finite cyclic.
2. For  $s \in \mathbb{Q} \cup \{\infty\}$ , the isotropy group  $\Gamma_s$  has finite index in the isotropy group  $\mathrm{SL}_2(\mathbb{Z})_s$ .

**Definition 2.13.**

1. Let  $\tau \in \mathcal{H}$ . The *period* of  $\tau$  is the number

$$h_\tau = |\pm \Gamma_\tau / \{\pm \mathrm{id}\}|$$

of maps in the isotropy group of  $\tau$ . The period of  $\tau$  only depends on the class  $\Gamma\tau$ . If  $h_\tau > 1$ , then we call the point  $\tau$ , or interchangeably the point  $\pi(\tau)$ , an *elliptic*

point for  $\Gamma$ . We further define the map  $\delta_\tau: \mathcal{H}^* \rightarrow \mathbb{C}$  to be the map represented by the matrix

$$\delta_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\bar{\tau} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

and the map  $\rho_\tau: \mathbb{C} \rightarrow \mathbb{C}$  by  $\rho_\tau(z) = z^{h_\tau}$ .

2. Let  $s \in \mathbb{Q} \cup \{\infty\}$ . The *width* of  $s$  is the number

$$h_s = [\mathrm{SL}_2(\mathbb{Z})_s : \pm\Gamma_\tau].$$

This also only depends on the class  $\Gamma s$ . We call the point  $s$ , or interchangeably the point  $\pi(s)$ , a *cusps* for  $\Gamma$ . Furthermore, we define  $\delta_s \in \mathrm{SL}_2(\mathbb{Z})$  to be any map taking  $\infty$  to  $s$ . Finally, define the map  $\rho_s(z): \mathcal{H}^* \rightarrow \mathbb{C}$  by  $\rho_s(z) = \exp(2\pi iz/h_s)$ . Note that this is well-defined on  $\infty$  since we restrict to the upper half-plane.

**Remark 2.14.** There are only two elliptic points on  $X(\mathrm{SL}_2(\mathbb{Z}))$ , namely the  $\mathrm{SL}_2(\mathbb{Z})$ -classes of  $i$  and  $\mu_3$ , where  $\mu_3 = \exp(2\pi i/3)$ . Their periods are 2 and 3, respectively. Since  $\Gamma_\tau \subseteq \mathrm{SL}_2(\mathbb{Z})_\tau$ , each elliptic point  $\tau$  for  $\Gamma$  has period 2 or 3, and lies in one of the classes  $\Gamma i$  or  $\Gamma \mu_3$ , according to whether its period is 2 or 3. Since  $\Gamma$  has finite index in  $\mathrm{SL}_2(\mathbb{Z})$ , it follows that there are only finitely many elliptic points on  $X(\Gamma)$ .

The only cusp of  $\mathrm{SL}_2(\mathbb{Z})$  is  $\infty$ , whose isotropy subgroup is the group generated by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Hence the width of  $\infty$  with respect to  $\Gamma$  is the smallest  $h$  such that  $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ , cf. Definition 2.5.

For  $\tau \in \mathcal{H}^*$ , let  $U_\tau$  be an open neighborhood of  $\tau$  such that for all  $\gamma \in \mathcal{H}$ , if  $\gamma(U_\tau)$  meets  $U_\tau$  then  $\gamma \in \Gamma_\tau$ , as per Corollary 2.10. If  $\tau$  is an elliptic point or a cusp for  $\Gamma$ , we may assume that  $U_\tau$  contains no further preimages of elliptic points or cusps. If  $\pi(\tau)$  is neither, we may assume that  $U_\tau$  contains no elliptic points or cusps altogether.

Define the map  $\varphi_\tau = \rho_\tau \circ \delta_\tau|_{U_\tau}$ . This is a map with open image in  $\mathbb{C}$ . For  $\tau_1, \tau_2 \in U_\tau$ , we have  $\varphi_\tau(\tau_1) = \varphi_\tau(\tau_2)$  if and only if  $\pi(\tau_1) = \pi(\tau_2)$ . Hence, the map  $\varphi_\tau$  induces an injective continuous map  $\psi_\tau: U_\tau \rightarrow \mathbb{C}$ . We take  $(\pi(U_\tau), \phi_\tau)$  to be our chosen chart around  $\pi(\tau)$ .

**Proposition 2.15.** *The charts  $(\pi(U_\tau), \psi_\tau)_{\tau \in \mathcal{H}^*}$  form a holomorphic atlas for the modular curve  $X(\Gamma)$ .*

**Proposition 2.16.** *Let  $\Gamma_1$  and  $\Gamma_2$  be two conjugation subgroups with  $\Gamma_1 \subseteq \Gamma_2$ . If  $-\mathrm{id} \in \Gamma_2 \setminus \Gamma_1$ , then the induced morphism  $X(\Gamma_1) \rightarrow X(\Gamma_2)$  has degree  $[\Gamma_2 : \Gamma_1]/2$ , else it has degree  $[\Gamma_2 : \Gamma_1]$ . The ramification index of a point  $\pi_1(\tau) \in X(\Gamma_1)$  is  $[\pm\Gamma_{2,\tau} : \pm\Gamma_{1,\tau}]$ .*

**Corollary 2.17.** *Let  $f: X(\Gamma) \rightarrow X(\mathrm{SL}_2(\mathbb{Z}))$  be the morphism induced by the inclusion  $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ . The ramification index of a point  $\pi(\tau) \in X(\Gamma)$  is  $h_\tau$ , if  $\tau$  is a cusp for  $\Gamma$  or if  $\tau$  is an elliptic point for  $\mathrm{SL}_2(\mathbb{Z})$  but not for  $\Gamma$ . Else, the ramification index is 1.*

Let  $y_2$ ,  $y_3$ , and  $y_\infty$  be the images of  $i$ ,  $\mu_3$ , and  $\infty$ , respectively, under the projection  $\mathcal{H}^* \rightarrow X(\mathrm{SL}_2(\mathbb{Z}))$ . For  $h \in \{2, 3\}$ , let  $\varepsilon_h$  denote the number of elliptic points in  $X(\Gamma)$  of period  $h$ . Further, denote by  $\varepsilon_\infty$  the number of cusps in  $X(\Gamma)$ .

The ramification locus of  $f$  is contained in the set  $y_1, y_2, y_\infty$ . Now let  $d$  be the degree of  $f$ . By applying the previous corollary and the formula

$$d = \sum_{x \in f^{-1}(y_h)} e_f(x),$$

one verifies the formulae

$$\sum_{x \in f^{-1}(y_h)} (e_f(x) - 1) = \frac{h-1}{h}(d - \varepsilon_h),$$

$$\sum_{x \in f^{-1}(\infty)} (e_f(x) - 1) = d - \varepsilon_\infty.$$

**Proposition 2.18.** *The genus  $g$  of the modular curve  $X(\Gamma)$  is given by*

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}.$$

*Proof.* The statement follows from the Riemann–Hurwitz formula, the previous discussion, and the fact that  $X(\mathrm{SL}_2(\mathbb{Z}))$  has genus 0, which will be proved later in Example ?? □

### 3. Basic facts and definitions

This section collects some basic facts and definitions that will be of use later in this work.

#### 3.1. Covering spaces

**Definition 3.1.** Let  $X$  be a topological space,  $F$  a set endowed with the discrete topology, and  $G$  a group acting on both  $X$  and  $F$ . Define the fibred product  $X \times_G F$  to be the topological space  $(X \times F) / \sim$ , where  $(x, f) \sim (gx, gf)$  for all  $g$  in  $G$ .

**Proposition 3.2.** *Let  $X$  be a connected, locally pathwise connected, and semi-locally simply connected topological space. Let  $p: \tilde{X} \rightarrow X$  be a universal cover. Furthermore, choose a point  $\tilde{x}_0$  of  $\tilde{X}$ , and let  $x_0$  be the image of  $\tilde{x}_0$  in  $X$ . Denote the fundamental group  $\pi_1(X, x_0)$  by  $\pi_1$ . Then there is an equivalence of categories*

$$\{\text{Unbranched covers of } X\} \longrightarrow \{\pi_1\text{-sets}\},$$

defined by the pair of quasi-inverse functors

$$(p_Y: Y \rightarrow X) \mapsto p_Y^{-1}(x_0) \quad \text{and} \quad F \mapsto \tilde{X} \times_{\pi_1} F.$$

*Proof.* One verifies by hand that the given functors are mutually quasi-inverse, by using elementary covering theory. Nonetheless, the needed isomorphisms between objects are given below.

Let  $F$  be a  $\pi_1$ -set and  $p_F: \tilde{X} \times_{\pi_1} F \rightarrow X$  the associated covering. Define a map  $\zeta_F: F \rightarrow p_F^{-1}(x_0)$  by sending an element  $f$  to the class of  $(\tilde{x}_0, f)$ .

On the other hand, let  $p_Y: Y \rightarrow X$  be a cover of  $X$ . Define a map

$$\eta_Y: \tilde{X} \times_{\pi_1} p_Y^{-1} \rightarrow Y$$

as follows. For a given class  $(\tilde{x}, f)$ , let  $\beta: [0, 1] \rightarrow \tilde{X}$  be a path starting in  $\tilde{x}_0$  and ending in  $\tilde{x}$ . Consider the projection  $p\beta$  of  $\beta$  to  $X$  and lift the path  $p\beta$  to a path  $\tilde{\beta}_f$  in  $Y$ , with starting point  $f$ . Finally, set  $\eta_Y(\tilde{x}, f) = \tilde{\beta}_f(1)$ . Note that since  $\tilde{X}$  is simply connected, this is independent of the choice of the path  $\beta$ . Also, the map is well-defined, since  $p\beta\tilde{\gamma} = p\beta$  for any lift  $\tilde{\gamma}$  of a loop in  $X$ .  $\square$

**Remark 3.3.** In the above proposition, if  $X$  has the structure of a Riemann surface, then the first category may be taken to be the category of unbranched covers of Riemann surfaces over  $X$ . Indeed, every cover inherits a complex structure from  $X$  such that the structure map becomes holomorphic, and morphisms of covers of  $X$  are automatically holomorphic. Indeed, if  $g: C' \rightarrow C$  is a continuous map and  $f: C \rightarrow X$  is an open and holomorphic map such that  $f \circ g$  is holomorphic, then  $g$  is holomorphic; see [Lam, 1.3.7].

Furthermore, let  $X$  be a Riemann surface, let  $S \subset X$  be a finite set. Then putting  $(C, p) \mapsto (C \setminus p^{-1}(S), p)$  defines an equivalence of categories between the category of finite covers of  $X$  with ramification locus contained in  $S$  and the category of finite unbranched covers of  $X \setminus S$ . The reason is roughly that the local data of an unbranched cover around a “missing” branch point uniquely characterizes that of any extension of that cover to a ramified one, e.g. the local degree of the cover map will correspond to the ramification index. The topic of extending unbranched covers to branched ones is discussed in detail in [Lam, 4.6].

### 3.2. Complex curves

**Proposition 3.4.** *The assignment  $C \mapsto K(C)$  defines a contravariant equivalence of categories between the category of irreducible smooth curves over  $\mathbb{C}$  and the category of finitely generated field extensions of  $\mathbb{C}$  of transcendence degree one. By definition, degree  $d$  maps of curves correspond to degree  $d$  field extensions.*

*Proof.* See [Sil09, pp.20-22] □

**Proposition 3.5** (Riemann–Hurwitz formula). *Let  $\varphi: C_1 \rightarrow C_2$  be a finite, degree  $d$  map of smooth curves of genera  $g_1$  and  $g_2$ , respectively. Then*

$$2g_1 - 2 = d(2g_2 - 2) + \sum_{x \in C_1} (e_\varphi(x) - 1),$$

where  $e_\varphi(x)$  is the ramification index of  $\varphi$  at  $x$ .

*Proof.* See [Sil09, Thm. 5.9] or [Lam, 7.2.1]. □

### 3.3. Further definitions

**Definition 3.6.** Let  $X$  be a set. A *weighting* on  $X$  is a function  $w: X \rightarrow [0, \infty]$ . For an element  $x$  of  $X$ , the value  $w(x)$  is called the *weight* of  $x$ . The *weighted count of the elements of  $X$*  is defined as the sum  $\sum_{x \in X} w(x)$ .

## 4. Covers of an elliptic curve

In this section we define the central notions and objects of interest, i. e. finite covers of an elliptic curve with simple ramification type, the weighted counts of isomorphism classes thereof, and the generating functions associated to such weighted counts.

In the following, let  $\mathbb{C}$  be the ground field for all varieties considered.

**Definition 4.1.** Let  $E$  be an elliptic curve.

1. A *cover* of  $E$  is a finite morphism  $p: C \rightarrow E$  of a disjoint union  $C = \cup_{i=1}^k C_i$  of  $k$  irreducible smooth curves  $C_i$ . We shall denote the genus of  $C$  by  $g$  and the degree of  $p$  by  $d$ . Often a cover will be referred to by its source  $C$ .
2. Let  $S = \{b_1, \dots, b_{2g-2}\}$  be a set of  $2g-2$  distinct points of  $E$ . A cover  $C$  of genus  $g$  is *simply branched over  $S$* , if it is simply branched over each point of  $S$ . This means that for all points  $b$  of  $S$  there is exactly one point  $x$  in  $p^{-1}(b)$  with ramification index  $e_p(x) = 2$ , the others having a ramification index one.

It follows from the Riemann-Hurwitz formula of Proposition 3.5 that for a simply branched cover  $C \rightarrow E$ , every point not in the pre-image of  $S$  has a ramification index one. This justifies the choice of the number of points in  $S$ .

3. Two covers  $C_1, C_2$  are to be considered isomorphic, if there is an isomorphism  $C_1 \rightarrow C_2$  commuting with the respective structure maps into  $E$ . Accordingly, define the automorphism group  $\text{Aut}(C)$  of the cover  $C$  to be the group of cover isomorphisms  $C \rightarrow C$ .
4. A *connected cover* is a cover with connected source  $C$ , i. e. with only one irreducible component.

**Remark 4.2.** Let  $C = C_1 \cup \dots \cup C_k$  be a cover of genus  $g$  with structure map  $p$  of degree  $d$ . For all  $i$ , let  $p_i$  be the connected cover defined by the restriction  $p|_{C_i}$ . Denote the genus of  $C_i$  by  $g_i$  and the degree of  $p_i$  by  $d_i$ . By the Riemann-Hurwitz formula, the maps  $p_i$  have  $2g_i - 2$  ramification points on  $C_i$ . Hence, the following relations hold:

$$\sum_i d_i = d, \text{ and } \sum_i (2g_i - 2) = 2g - 2.$$

**Proposition 4.3.** *Let  $C$  be a connected cover of  $E$ . Then the automorphism group of  $C$  is finite.*

*Proof.* By Proposition 3.4, if  $C$  is a connected cover of  $E$ , then the elements of  $\text{Aut}(C)$  correspond to the automorphisms of the finite field extension  $K(C)/K(E)$ , of which only finitely many exist.  $\square$

**Proposition 4.4.** *Let  $C = C_1 \cup \dots \cup C_k$  be a cover, and  $p_i := p|_{C_i}$ . Then the automorphism group of  $C$  is given by the semidirect product*

$$\mathrm{Aut}(C) = \prod_i \mathrm{Aut}(C_i) \rtimes \Gamma,$$

where  $\Gamma \subset \mathrm{Sym}\{C_1, \dots, C_k\}$  is the subgroup generated by the automorphisms that permute isomorphic components. In particular,  $\mathrm{Aut}(C)$  is finite.

*Proof.* The map  $\mathrm{Aut}(C) \rightarrow \Gamma$  given by looking at the action of an automorphism on the set  $\{C_1, \dots, C_k\}$  is part of a short exact sequence

$$1 \longrightarrow \prod_i \mathrm{Aut}(C_i) \longrightarrow \mathrm{Aut}(C) \longrightarrow \Gamma \longrightarrow 1$$

which admits a splitting  $\Gamma \rightarrow \mathrm{Aut}(C)$  given by the inclusion.  $\square$

**Remark 4.5.** If the cover  $C$  is simply branched over  $S$ , then no two components of genus greater than one are isomorphic as connected covers, since any isomorphism would have to preserve ramification indices (see for example [Sil09, II, Prop. 2.6]), but no two components share a branched point over  $E$ . In particular, if there are no components of genus one, then  $\Gamma = \{1\}$ .

On the other hand, each component of genus one is unramified over  $E$ , and could be isomorphic to other components of genus one, in which case  $\Gamma$  is nontrivial.

**Definition 4.6.** Let  $E$  be an elliptic curve,  $S = \{b_1, \dots, b_{2g-2}\}$  a set of  $2g - 2$  distinct points of  $E$ .

1. Let  $\mathrm{Cov}(E, S)_{g,d}$  be the set of isomorphism classes of covers of  $E$  of genus  $g$  and degree  $d$  that are simply branched over  $S$ .
2. Any isomorphism of two equivalent covers defines a bijection of their automorphism groups. This allows one to define the *weight* of the class  $[C]$  to be the number  $1/|\mathrm{Aut}(C)|$ .
3. Define  $\widehat{N}_{g,d}$  to be the weighted count

$$\widehat{N}_{g,d} := \sum_{C \in \mathrm{Cov}(E, S)_{g,d}} \frac{1}{|\mathrm{Aut}(C)|}$$

of the (classes of) covers of  $E$ .

4. Let  $\mathrm{Cov}(E, S)_{g,d}^\circ \subset \mathrm{Cov}(E, S)_{g,d}$  be the subset of classes  $[C]$  such that  $C$  is connected.
5. Similarly, define  $N_{g,d}$  to be the the weighted count

$$N_{g,d} := \sum_{C \in \mathrm{Cov}(E, S)_{g,d}^\circ} \frac{1}{|\mathrm{Aut}(C)|}$$

of the connected covers of  $E$ .



To shorten the notation, the elliptic curve  $E$  and the set of points  $S$  are omitted from the notation. It will turn out that  $\widehat{N}_{g,d}$  and  $N_{g,d}$  are finite and do not depend on the choice of  $E$  and  $S$ .

**Definition 4.7.** For any  $g \geq 1$ , define  $F_g$  to be the generating series

$$F_g(q) = \sum_{d \geq 1} N_{g,d} q^d$$

counting connected covers of genus  $g$ .

**Example 4.8.** By the theory of elliptic curves,  $N_{1,d} = \sum_{j|d} 1/j$ . Indeed, Let  $E$  be defined by the lattice  $\Omega = \langle 1, i \rangle$ . The set of divisors of  $d$  classify the covers of  $E$  by assigning to some  $j|d$  the elliptic curve  $C_j$  defined by  $\Omega_j = \langle 1, di/j^2 \rangle$  and the cover map  $C_j \rightarrow E: z \mapsto jz$ . There are  $j$  automorphisms  $C_j \rightarrow C_j$  of this cover, given by  $z \mapsto z + k/j, k = 0, 1, \dots, j-1$ .

Further, by using the power series expansion for the logarithm

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$$

for  $|z| < 1$ , one finds that  $-\sum_{n \geq 1} \log(1 - q^n) = \sum_{d \geq 1} \sum_{j|d} \frac{1}{j} q^d$ . Hence, the first generating function is given by

$$F_1(q) = -\sum_{n \geq 1} \log(1 - q^n).$$

This thesis shall prove the following result.

**Theorem 4.9** ([Dij]). *Let  $g \geq 2$ , and for  $\tau \in \mathbb{C}$  let  $q(\tau) = \exp(2\pi i\tau)$ . Then the function  $F_g(q)$  is a quasimodular form of weight  $6g - 6$ .*

The strategy to prove the theorem will involve considering a more general generating function counting all covers of genus  $g$  and degree  $d$ . This generating function will be easier to compute.

**Definition 4.10.** The generating functions  $Z(q, \lambda)$  and  $\widehat{Z}(q, \lambda)$  for  $N_{g,d}$  and  $\widehat{N}_{g,d}$  respectively, are defined as follows:

$$Z(q, \lambda) := \sum_{g \geq 1} \sum_{d \geq 1} \frac{N_{g,d}}{(2g-2)!} q^d \lambda^{2g-2} = \sum_{g \geq 1} \frac{F_g(q)}{(2g-2)!} \lambda^{2g-2},$$

$$\widehat{Z}(q, \lambda) := \sum_{g \geq 1} \sum_{d \geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!} q^d \lambda^{2g-2}.$$

**Lemma 4.11.** *The above generating functions satisfy the relation*

$$\widehat{Z}(q, \lambda) = \exp(Z(q, \lambda)) - 1.$$

*Proof.* The proof is subdivided into three parts. First, some notation and terminology is introduced. Second, the coefficient of  $q^d \lambda^{2g-2}$  in  $\exp(Z(q, \lambda)) - 1$  is expressed in terms of the new notation. Third, combinatorial arguments are used to prove that this coefficient is equal to  $\widehat{N}_{g,d}/(2g-2)!$ .

Let  $C$  be a degree  $d$ , genus  $g$  cover. The *combinatorial type* of  $C$  is the tuple  $\kappa = (k_j, g_j, d_j)_{j=1}^r$  of natural numbers, such that for each  $j$ , the space  $C$  contains exactly  $k_j$  connected components  $C_j$  of genus  $g_j$  such that the cover map  $C_j \rightarrow E$  is of degree  $d_j$ . For simplicity, denote the Euler characteristics  $2g-2$  and  $2g_j-2$  by  $\chi$  and  $\chi_j$ , respectively. Then

$$\sum_j d_j = d, \text{ and } \sum_j \chi_j = \chi.$$

Further, define  $\widehat{N}_\kappa$  to be the weighted count of the covers of combinatorial type  $\kappa$ . Then

$$\widehat{N}_{g,d} = \sum_{|\kappa|=(\chi,d)} \widehat{N}_\kappa,$$

where  $|\kappa|$  is defined as the tuple  $(\sum_j k_j \chi_j, \sum_j k_j d_j)$ , for  $\kappa = (k_j, g_j, d_j)_j$ . Finally, note that the relation

$$q^d \lambda^\chi = \prod_{j=1}^r q^{k_j d_j} \lambda^{k_j \chi_j}$$

holds for each  $\kappa = (k_j, g_j, d_j)_j$  with  $|\kappa| = (\chi, d)$ .

The exponential of  $Z(q, \lambda)$  is given by

$$\exp(Z(q, \lambda)) = \prod_{g \geq 1} \prod_{d \geq 1} \sum_{k \geq 0} \frac{N_{g,d}^k}{k! (\chi!)^k} q^{kd} \lambda^{k\chi}.$$

Expanding, one finds that the expression for  $\exp(Z(q, \lambda))$  is a sum over terms of the form

$$\prod_{j=1}^{<\infty} \left( \frac{N_{g_j, d_j}}{\chi_j!} \right)^{k_j} \frac{1}{k_j!} q^{k_j d_j} \lambda^{k_j \chi_j},$$

for some choices of parameters  $g_j, d_j, k_j$ . Such choices may be collected to form combinatorial types  $\kappa = (g_j, d_j, k_j)_j$ . Now, by collecting the summands arising from choices that induce combinatorial types of the same absolute value  $|\kappa|$ , one obtains that the coefficient of  $q^d \lambda^\chi$  in  $\exp(Z(q, \lambda))$  is equal to the sum  $\sum_{|\kappa|=(\chi,d)} a_\kappa$ , where

$$a_\kappa = \prod_{j=1}^r \left( \frac{N_{g_j, d_j}}{\chi_j!} \right)^{k_j} \frac{1}{k_j!}.$$

It remains to prove that  $a_\kappa = \widehat{N}_\kappa$  for each combinatorial type  $\kappa$ . Sketch:

- The product  $N_{g_1, d_1}^{k_1} \cdots N_{g_r, d_r}^{k_r}$  represents a choice of connected components of a cover. One has to modify this product to account for the automorphisms of the components and the choice of ramification points.

- There are  $\binom{x}{\chi_1, \chi_1, \dots, \chi_r} = x \prod_{j=1}^r 1/(\chi_j!)^{k_j}$  ways to subdivide  $S$  into subsets that serve as the ramification locus of the connected components. Here, each  $\chi_j$  appears in the binomial coefficient  $k_j$  times.
- For components of genus  $\geq 2$ , we get a factor of  $1/k_j!$  to account for overcounting.
- For components of genus 1 we also get a factor of  $1/k_j!$ , since we either overcount (if some genus 1 covers are non-isomorphic) or get extra automorphisms as permutations (if some genus 1 covers are isomorphic).
- Putting this all together, we get that the weighted count of covers of type  $\kappa$  is  $a_\kappa$ .

□

## 5. Classifying covers via the fundamental group

Let  $E$  be an elliptic curve,  $S = \{b_1, \dots, b_{2g-2}\}$  a set of  $2g-2$  distinct points of  $E$ . Fix a basis point  $b_0 \in E \setminus S$ , and denote the fundamental group  $\pi_1(E \setminus S, b_0)$  by  $\pi_1$ . Recall the equivalence of categories from 3.1.:

$$\left\{ \begin{array}{l} \text{Finite ramified covers of } E \\ \text{with ramification locus contained in } S \end{array} \right\} \longrightarrow \{\pi_1\text{-sets}\}.$$

The goal of this section is to use this equivalence of categories to classify those  $\pi_1$ -sets giving rise to unbranched covers that, after adding the branched points, become the covers we are interested in, i.e. the over  $S$  simply branched, genus  $g$ , degree  $d$  covers. Note that a simply branched cover of genus  $g$  is ramified over exactly  $2g-2$  points of  $E$ . It follows that if its ramification locus  $S_0$  is contained in  $S$ , then  $S_0 = S$ .

To obtain natural  $\pi_1$ -actions on the set of  $d$  fibre points of  $b_0$ , it is convenient to introduce markings on the set of fibres.

### 5.1. Marked covers and the monodromy map

**Definition 5.1.** A *marked* (degree  $d$ , genus  $g$ , simply branched over  $S$ ) cover of  $E$  is a triple  $(C, p, m)$ , where  $(C, p) \in \text{Cov}(E, S)_{g,d}$  and  $m: p^{-1}(b_0) \rightarrow \{1, \dots, d\}$  is a bijective map, the *marking* of  $(C, p, m)$ .

Two marked covers  $(C_1, p_1, m_1)$  and  $(C_2, p_2, m_2)$  are considered equivalent, if there is an isomorphism of covers  $\phi: C_1 \rightarrow C_2$  such that  $m_1 = m_2 \phi$ . Let  $\widetilde{\text{Cov}}(E, S)_{g,d}$  denote the set of equivalence classes of marked covers with respect to this relation.

**Definition 5.2.** Let  $(C, p)$  be a cover of  $E$ . Denote the group operation of  $\pi_1$  on the fibre of  $p^{-1}(b_0)$  by  $(\gamma, x) \mapsto \gamma \cdot x$ . Define the monodromy map

$$\text{mon}: \widetilde{\text{Cov}}(E, S)_{g,d} \rightarrow \text{Hom}(\pi_1, S_d)$$

by  $\text{mon}(C, p, m)(\gamma)(i) = m(\gamma \cdot m^{-1}(i))$ .

Let the symmetric group  $S_d$  operate on the first set by  $\sigma \cdot (C, p, m) = (C, p, \sigma m)$ , and on the second by  $\sigma \cdot \psi = \text{inn}(\sigma)\psi$ , i.e. by inner automorphisms. Then  $\text{mon}$  becomes a morphism of  $S_d$ -sets. Furthermore, for an element  $\psi = \text{mon}(C, p, m)$  of the image of  $\text{mon}$ , the group action “forgetting the marking”

$$m^{-1}\psi(\_)m: \pi_1 \rightarrow \text{Aut}(p^{-1}(b_0))$$

on the fiber of  $b_0$  is the same as the one defined by the above equivalence of categories.

**Definition 5.3.** The  $S_d$ -set  $\widehat{T}_{g,d}$  is defined by

$$\widehat{T}_{g,d} = \{(\tau_1, \dots, \tau_{2g-2}, \sigma_1, \sigma_2) \in S_d^{2g}; \text{ each } \tau_i \text{ is a simple transposition,} \\ \tau_1 \cdots \tau_{2g-2} = \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1}\},$$

where the  $S_d$ -action is defined by conjugation in each component, after noting that conjugates of transpositions are transpositions.

**Proposition 5.4.** *The image of mon is isomorphic as a  $S_d$ -set to  $\widehat{T}_{g,d}$ .*

*Proof.* The fundamental group  $\pi_1$  of  $E \setminus S$  is described by the following generating set and relation:

$$\pi_1 = \langle \gamma_1, \dots, \gamma_{2g-2}, \alpha_1, \alpha_2; \gamma_1 \cdots \gamma_{2g-2} = \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \rangle.$$

For over  $S$  simply branched covers, the image of each loop  $\gamma_i$  under the monodromy map is a simple transposition  $\tau_i$ . Namely, there is over  $b_i$  exactly one branch point of index 2, and  $\tau_i$  interchanges the two fiber points corresponding to the two sheets of the branching, leaving the other fiber points unchanged.

Combining these remarks, one finds that putting

$$\psi \mapsto (\psi(\gamma_1), \dots, \psi(\gamma_{2g-2}), \psi(\alpha_1), \psi(\alpha_2))$$

defines the required isomorphism, which is compatible with the  $S_d$ -action.  $\square$

**Proposition 5.5.** *The morphism of  $S_d$ -sets  $\rho: \widetilde{\text{Cov}}(E, S)_{g,d} \rightarrow \widehat{T}_{g,d}$  induces a bijection on the sets of orbits*

$$S_d \backslash \widetilde{\text{Cov}}(E, S)_{g,d} \rightarrow S_d \backslash \widehat{T}_{g,d}.$$

*Proof.* To see that  $\rho$  is surjective, let  $t \in \widehat{T}_{g,d}$ , and let  $\psi_t: \pi_1 \rightarrow S_d$  be the corresponding group homomorphism. By the above equivalence of categories, the  $\pi_1$ -action on  $\{1, \dots, d\}$  defined by  $\psi_t$  gives a finite, unbranched cover of Riemann surfaces  $C' \rightarrow E \setminus S$ , which may be extended to a branched cover  $C \rightarrow E$ , see the remark in 3.1.. The  $\pi_1$ -action on  $\{1, \dots, d\}$  gives the  $\pi_1$ -action on the fiber of the basis point  $b_0$  associated to  $(C, p)$ , showing that the extension  $C$  has the right branching.

For the injectivity on the sets of orbits, let  $\rho(C_1, p_1, m_1) = t$  and  $\rho(C_2, p_2, m_2) = \sigma \cdot \psi_t$ , for some  $t \in \widehat{T}_{g,d}$  and  $\sigma \in S_d$ . Then  $\rho(C_2, p_2, \sigma^{-1}m_2) = t$ . Let  $\psi_t$  define the associated group action on  $\{1, \dots, d\}$ , hence the group action on the fibers. From the equivalence of categories follows that the two marked covers differ only by the marking:  $C_1 \simeq C_2$ . Hence, the two marked covers are in the same orbit.  $\square$

**Remark 5.6.** The  $S_d$ -orbits of  $\widetilde{\text{Cov}}(E, S)_{g,d}$  are in one-to-one correspondence with the elements of  $\text{Cov}(E, S)_{g,d}$ . The above proposition gives thus a bijection of  $\text{Cov}(E, S)_{g,d}$  with the set of  $S_d$ -orbits of  $\widehat{T}_{g,d}$ .

## 5.2. Counting covers

By the above discussion, we get an algebraic description of the weighted count  $\widehat{N}_{g,d}$  of genus  $g$ , degree  $d$ , simply branched over  $S$ , covers of  $E$ .

**Proposition 5.7.** *Let  $(C, p, m)$  be a marked cover and  $t$  its image under  $\rho$ . Then there is a group isomorphism  $\text{Aut}_p(C) \rightarrow \text{Stab}(t)$ .*

*Proof.* Let  $\phi_t$  be the group homomorphism  $\pi_1 \rightarrow S_d$  corresponding to  $t$ . By the equivalence of categories,  $\text{Aut}_p(C)$  is isomorphic to the group of automorphisms of the  $\pi_1$ -action on  $\{1, \dots, d\}$  defined by  $\psi_t$ , i.e. those elements  $\sigma$  in the symmetry group  $S_d$  commuting with  $\psi_t$ , i.e. such that  $\psi_t = \text{inn}(\sigma)\psi_t$ . This condition translates under the isomorphism of  $S_d$ -sets in 5.4  $\square$

**Lemma 5.8.** *The following equality for the weighted count  $\widehat{N}_{g,d}$  holds:*

$$\widehat{N}_{g,d} = |\widehat{T}_{g,d}|/d!.$$

*Proof.* By propositions 5.5 and 5.7, the weighted count  $\widehat{N}_{g,d}$  is equal to the weighted count of the  $S_d$ -orbits of  $\widehat{T}_{g,d}$ , where each orbit is weighted by  $1/|\text{Stab}(t)|$ , for any element  $t$  in the orbit (this is well-defined since elements of the same orbits have isomorphic stabilizer subgroups). Now, it follows from the formula  $|\text{Orb}(t)| = |S_d|/|\text{Stab}(t)|$  that this weighted count equals  $|\widehat{T}_{g,d}|/d!$ .  $\square$

## 6. Conjugacy classes of the symmetric group

In this section, we further the computation of  $\widehat{N}_{g,d}$  by using similar techniques to the one applied when counting cycles in a graph. The rough picture is one of a graph with vertices the conjugacy classes of  $S_d$  and edges representing the passage from one class to another by multiplication with a simple transposition. We seek to count not cycles, but cycles starting and ending with the same representative, in the sense specified in the section. To do this, we make use of an analog of the adjacency matrix for a graph.

To abbreviate, we use the term “transposition” for simple transpositions.

### 6.1. Conjugacy cycles

Recall the definition

$$\widehat{T}_{g,d} = \{(\tau_1, \dots, \tau_{2g-2}, \sigma_1, \sigma_2) \in S_d^{2g}; \text{ each } \tau_i \text{ is a transposition, } \tau_1 \cdots \tau_{2g-2} = \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1}\}.$$

Our aim is now to rewrite this definition using conjugacy classes. Note that the condition in the definition is equivalent to

$$(\tau_1 \cdots \tau_{2g-2})\sigma_2 = \sigma_1 \sigma_2 \sigma_1^{-1}. \quad (1)$$

**Definition 6.1.** For  $\sigma_2 \in S_d$ , define

$$P_{g,d}(\sigma_2) = \{(\tau_1, \dots, \tau_{2g-2}) \in S_d^{2g-2}; \text{ each } \tau_i \text{ is a transposition, } \tau_1 \cdots \tau_{2g-2} \sigma_2 \text{ is conjugate to } \sigma_2\}.$$

If  $g = 1$ , define  $P_{g,d}$  to be the singleton set  $\{\bullet\}$ . Further, let  $c(\sigma_2)$  denote the conjugacy class of  $\sigma_2$ .

**Proposition 6.2.** Let  $\mathcal{R} = (\sigma_2^{(1)}, \dots, \sigma_2^{(r)})$  be a system of (distinct) representatives of the conjugacy classes of  $S_d$ . Then

$$|\widehat{T}_{g,d}| = \sum_{\sigma_2 \in \mathcal{R}} d! |P_{g,d}(\sigma_2)|.$$

*Proof.* Let  $\sigma_2 \in S_d$ , let  $(\tau_1, \dots, \tau_{2g-2}) \in P_{g,d}(\sigma_2)$  and let  $\sigma_1$  be an element such that  $(\tau_1 \cdots \tau_{2g-2})\sigma_2 = \sigma_1 \sigma_2 (\sigma_1^{-1})$ . Then there is a bijection of the set of elements  $\sigma_1$  satisfying (1) onto the set of elements commuting with  $\sigma_2$ , given by sending  $\sigma_1$  to  $(\sigma_1^{-1})\sigma_1$ . The number of elements commuting with  $\sigma_2$  is given by the cardinality of the stabilizer  $|\text{Stab}(\sigma_2)| = |S_d|/|c(\sigma_2)| = d!/|c(\sigma_2)|$ . Thus, one obtains

$$|\widehat{T}_{g,d}| = \sum_{\sigma \in S_d} \frac{d!}{|c(\sigma)|} |P_{g,d}(\sigma)|.$$

Further, the function  $|P_{g,d}| : S_d \rightarrow \mathbb{C}$  is constant on conjugacy classes. Indeed, for  $\sigma \in S_d$  there is a bijection of  $P_{g,d}(\sigma_2)$  onto  $P_{g,d}(\sigma\sigma_2\sigma^{-1})$  given by conjugation with  $\sigma$  in each component. From this follows the required equality.  $\square$

**Corollary 6.3.** *The above proposition, together with Lemma 5.8, give the equality*

$$\widehat{N}_{g,d} = \sum_{\sigma_2 \in \mathcal{R}} |P_{g,c}(\sigma_2)|.$$

From now on, let  $\mathcal{R} = (\sigma_2^{(1)}, \dots, \sigma_2^{(r)})$  be a fixed system of representatives of the conjugacy classes of  $S_d$ . Then the cardinality  $r = \text{part}(d)$  of  $\mathcal{R}$  is the number of (unordered) partitions of  $\{1, \dots, d\}$ . This follows essentially from the fact that conjugation with a permutation acts on cycles by applying the permutation to the entries of the cycle.

## 6.2. Adjacency matrices

**Definition 6.4.** Let  $d \geq 1$  and  $k \geq 0$ .

1. For  $1 \leq i, j \leq r$ , define the sets  $N_{d,i,j}^k$  by

$$N_{d,i,j}^k = \{(\tau_1, \dots, \tau_k) \in S_d^k; \text{ each } \tau_i \text{ is a transposition,} \\ \tau_1 \cdots \tau_k \sigma_2^{(i)} \in c(\sigma_2^{(j)})\}.$$

For  $k = 0$ , define  $N_{d,i,j}^0 = \delta_{i,j}$  (Kronecker delta).

2. Define the size  $r$  square matrix  $M_d$  by

$$(M_d)_{i,j} = |N_{d,i,j}^1|.$$

This does not depend on the choice of system of representatives  $\mathcal{R}$ .

**Remark 6.5.** If  $k$  is odd, applying the signum homomorphism to the defining condition shows that  $N_{d,i,i}^k$  is empty. If  $k = 2g - 2$  is even, then  $N_{d,i,i}^{2g-2} = P_{g,d}(\sigma_2^{(i)})$ .

**Proposition 6.6.** *The entries of  $M_d^k$  are given by  $(M_d^k)_{i,j} = |N_{d,i,j}^k|$ .*

*Proof.* The proof is by induction on  $k$ . For  $k = 0, 1$ , there is nothing to show. For the induction step, note that if  $i$  (resp.  $j$ ) are fixed, the sets  $N_{d,i,j}^k$  are pairwise disjoint for varying  $j$  (reps.  $i$ ). Now define a function

$$\prod_{l=1}^r N_{d,i,l}^k \times N_{d,l,j}^1 \rightarrow N_{d,i,j}^{k+1}$$

as follows: for a given element  $((\tau_1, \dots, \tau_k), \tau_0)$ , let  $\sigma \in S_d$  be the unique element such that  $\tau_1 \cdots \tau_k \sigma_2^{(i)} = \sigma \sigma_2^{(l)} \sigma^{-1}$ , and define the image of  $((\tau_1, \dots, \tau_k), \tau_0)$  to



be  $(\sigma\tau_0\sigma^{-1}, \tau_1, \dots, \tau_k)$ . By the definition of matrix multiplication, it suffices to prove that this function is a bijection.

Injectivity is clear by the uniqueness of  $\sigma$  in the definition. For surjectivity, given an element  $(\tau_0, \tau_1, \dots, \tau_k)$  in the target, choose an  $l$  such that  $\tau_1 \cdots \tau_k \sigma_2^{(i)}$  is conjugate to  $\sigma_2^{(l)}$ , say  $\tau_1 \cdots \tau_k \sigma_2^{(i)} = \sigma_2^{(l)} \sigma^{-1}$ . Then  $(\sigma^{-1} \tau_0 \sigma) \sigma_2^{(l)}$  is conjugate to  $\sigma_2^{(j)}$ .  $\square$

**Lemma 6.7.** *Let  $d \geq 1$  and  $r = \text{part}(d)$ . Let  $\mu_{1,d}, \dots, \mu_{r,d}$  be the eigenvalues of  $M_d$ , listed according to their algebraic multiplicities. Then*

$$\widehat{Z}(q, \lambda) = \sum_{d \geq 1} \sum_{i=1}^r \exp(\mu_{i,d} \lambda) q^d.$$

*Proof.* Recall the definition of  $\widehat{Z}$ :

$$\widehat{Z}(q, \lambda) = \sum_{g \geq 1} \sum_{d \geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!} q^d \lambda^{2g-2}.$$

The above proposition and remark give  $(M_d^{2g-2})_{i,i} = |P_{g,d}(\sigma_2^{(i)})|$  and  $(M_d^k)_{i,i} = 0$  if  $k$  is odd, for all  $i$ . Hence, by 6.3 one has  $\widehat{N}_{g,d} = \text{Tr}(M_d^{2g-2}) = \sum_{i=1}^r \mu_{i,d}^{2g-2}$ , and since the terms for  $k$  odd vanish,

$$\begin{aligned} \widehat{Z}(q, \lambda) &= \sum_{g \geq 1} \sum_{d \geq 1} \frac{\text{Tr}(M_d^{2g-2})}{(2g-2)!} q^d \lambda^{2g-2} \\ &= \sum_{d \geq 1} \sum_{i=1}^r \sum_{g \geq 1} \frac{\mu_{i,d}^{2g-2}}{(2g-2)!} \lambda^{2g-2} q^d \\ &= \sum_{d \geq 1} \sum_{i=1}^r \exp(\mu_{i,d} \lambda) q^d. \end{aligned}$$

$\square$

## 7. The group algebra of the symmetric group

Let  $\mathbb{C}[S_d]$  be the group algebra of the symmetric group, let  $\mathcal{Z}_d$  be its centre. This is a commutative algebra, acting on itself linearly by multiplication. In this section, we relate this linear action to the matrix  $M_d$  of the previous section, and we use the representation and character theory of the symmetric group to compute its eigenvalues.

### 7.1. The centre of the group algebra

**Definition 7.1.** Let  $\mathcal{Z}_d \subset \mathbb{C}[S_d]$  be the centre of the group algebra. If  $c$  is a conjugacy class of  $S_d$ , define the element  $z_c \in \mathcal{Z}_d$  by

$$z_c = \sum_{\sigma \in c} \sigma.$$

**Remark 7.2.** The elements  $z_c$  lie in the centre since  $\alpha c = c\alpha$  for all conjugacy classes  $c$  and elements  $\alpha$  of  $S_d$ . Further, the  $z_c$  form a basis of  $\mathcal{Z}_d$ . Indeed, linear independence follows from the linear independence of the distinct elements  $\sigma \in S_d \subset \mathbb{C}[S_d]$ . Further, if  $z \in \mathcal{Z}_d$ , then the equalities  $\alpha z \alpha^{-1} = z$  show that the  $\mathbb{C}$ -coefficients of elements in the same conjugacy class are equal. Hence  $\mathcal{Z}_d$  is  $r$ -dimensional, with  $r = \text{part}(d)$ .

Recall the definition of  $M_d$  from the previous section. There, we fixed a system of representatives for the equivalence classes of  $S_d$ . However, since the definition does not depend from the chosen representatives, we may also define  $M_d$  to be a matrix indicised by the conjugacy classes of  $S_d$ , ordered in the same way as before. The new, equivalent definition is as follows.

**Definition 7.3.** Let  $c', c$  be conjugacy classes of  $S_d$ . Define the matrix  $M_d$  by

$$(M_d)_{c',c} = |\{\tau; \tau \text{ is a transposition such that } \tau\sigma_2 \in c'\}|,$$

where  $\sigma_2$  is any representative of  $c$ .

From now on, we choose the ordering of the basis  $\{z_c\}_c$  and the ordering of the columns of  $M_d$  to be compatible, i.e. coming from the same fixed ordering of the conjugacy classes  $\{c\}$ .

**Proposition 7.4.** *Let  $t$  be the conjugation class containing all transpositions,  $z_t$  the corresponding basis element of  $\mathcal{Z}_d$ . Let  $M_t$  be the size  $r$  square matrix representing the  $\mathbb{C}$ -linear map  $(z_t \cdot) : \mathcal{Z}_d \rightarrow \mathcal{Z}_d$  given by multiplication with  $z_t$ . Then  $M_t = (M_d)^\top$ .*

*Proof.* Let  $c, c'$  be conjugacy classes. Note that if  $z = \sum_{\sigma \in S_d} \lambda_\sigma \sigma = \sum_{c''} \lambda_{c''} z_{c''}$ , then the coefficient  $\lambda_{c''}$  is equal to the coefficient  $\lambda_{\sigma_2}$ , for any  $\sigma_2 \in c''$ . Now let  $\sigma_2 \in c$ , and consider the product

$$z_t z_{c'} = \left( \sum_{\tau \in t} \tau \right) \left( \sum_{\sigma' \in c'} \sigma' \right) = \sum_{\sigma \in S_d} \left( \sum_{\tau \sigma' = \sigma} 1 \right) \sigma.$$

In this expansion, the coefficient  $\lambda_{\sigma_2}$  of any element  $\sigma_2 \in c$  is the quantity  $|\{\tau \in t; \tau^{-1} \sigma_2 \in c'\}|$ . It follows that  $(M_t)_{c'; c} = \lambda_c = \lambda_{\sigma_2} = (M, d)_{c, c'}$ .  $\square$

## 7.2. Irreducible characters of the symmetric group

We have reduced our problem of computing the eigenvalues of  $M_d$  to the computation of the eigenvalues of  $M_t$ . More generally, we find that  $\mathcal{Z}_d$  actually has a basis  $\{w_\chi\}$ , indexed by the irreducible characters of  $S_d$ , such that each  $w_\chi$  is an eigenvector for all linear maps defined by multiplication with any element of  $\mathcal{Z}_d$ , and such that the corresponding eigenvalues are easy to compute.

- Definition 7.5.** 1. Let  $\rho$  be an irreducible representation of  $\mathbb{C}[S_d]$ , i.e. a group homomorphism  $\rho: S_d \rightarrow \text{GL}(\mathbb{C}^n)$  such that for each  $\sigma \in S_d$  there are no  $\rho(\sigma)$ -invariant subspaces. The *irreducible character associated to  $\rho$*  is defined as the map  $\chi_\rho: S_d \rightarrow \mathbb{C}$ ,  $\sigma \mapsto \text{Tr}(\rho(\sigma))$
2. An *irreducible character* of  $S_d$  is a map  $\chi: S_d \rightarrow \mathbb{C}$  of the form  $\chi = \chi_\rho$  for some irreducible representation  $\rho$ . Its *dimension*  $\dim(\chi)$  is defined as the dimension of the associated representation  $\dim \rho = \chi(1)$ .

For brevity, we will refer to irreducible characters simply as characters.

**Remark 7.6.** Characters are constant on conjugacy classes. It is therefore justified to write  $\chi(c) \in \mathbb{C}$  for a character  $\chi$  and a conjugacy class  $c$ .

**Remark 7.7.** The number of irreducible representations of a finite group, up to isomorphism, is equal to the number of its conjugacy classes (see for example [Serb], p. 19, Thm. 7). In the case of the symmetric group, both the set of conjugacy classes and the set of irreducible representations are indexed by the set of Young diagrams, in a natural way. The irreducible representations are recovered from the Young diagrams via Specht modules.

**Proposition 7.8.** *Let  $\chi, \chi'$  be characters. Then*

$$\sum_{\sigma \in S_d} \chi(\sigma) \chi'(\sigma^{-1} \sigma_1) = \begin{cases} \frac{d!}{\dim(\chi)} \chi(\sigma_1) & \text{if } \chi = \chi' \\ 0 & \text{else.} \end{cases}$$

*Further, if  $c, c'$ , then*

$$\sum_{\chi} \chi(c) \chi(c') = \begin{cases} \frac{d!}{|c|} & \text{if } c = c' \\ 0 & \text{else,} \end{cases}$$

*where the  $\chi$  runs through the irreducible characters of  $S_d$ .*

*Proof.* ... □

**Definition 7.9.** Let  $\chi$  be a character of  $S_d$ . Define the element  $w_\chi \in \mathcal{Z}_d$  by

$$w_\chi = \frac{\dim(\chi)}{d!} \sum_c \chi(c^{-1})z_c = \frac{\dim(\chi)}{d!} \sum_{\sigma \in S_d} \chi(\sigma^{-1})\sigma.$$

**Proposition 7.10.** *The  $w_\chi$  form a basis of  $\mathcal{Z}_d$ . With respect to this basis, if  $z = \sum_\chi a_\chi w_\chi$  is any element of  $\mathcal{Z}_d$ , then the linear map  $(z \cdot)$  is represented by the matrix  $\text{Diag}((a_\chi)_\chi)$ . With this notation, if  $z = z_t$ , then  $a_\chi = \binom{d}{2} \chi(t) / \dim(\chi)$ .*

*Proof.* The two formulae in the above proposition lead to the formulae

$$w_\chi w_{\chi'} = \begin{cases} w_\chi & \text{if } \chi = \chi' \\ 0 & \text{else} \end{cases} \quad (1)$$

and

$$z_c = \sum_\chi \left( \frac{|c^{-1}| \chi(c^{-1})}{\dim(\chi)} \right) w_\chi \quad (2)$$

respectively. By (1), the  $w_\chi$  are linearly independent (multiply a linear relation with one of the  $w_\chi$ ), and by (2) they span  $\mathcal{Z}_d$ . The second statement follows directly from (1). The last statement follows with (2) from  $t = t^{-1}$  and  $|t| = \binom{d}{2}$ . □

**Lemma 7.11.** *The eigenvalues of  $M_d$  are given by*

$$\mu_{i,d} = \frac{\binom{d}{2} \chi(t)}{\dim(\chi)},$$

where  $\chi$  is the  $i$ -th character and  $t$  is the conjugation class of  $S_d$  containing all transpositions.

*Proof.* By proposition 7.4, the eigenvalues of  $M_d$  are the same as the eigenvalues of  $M_t$ . Now the statement follows from the second and third statements of the above proposition, since the matrix  $M_t$  represents multiplication with  $z_t$ . □

## 8. Subsets of the half integers

In this section, we use a formula of Frobenius to express the function  $\widehat{Z}_{(q,\lambda)}$  as the constant term of a certain product of Laurent series. This is exactly what is needed in the next section to prove that  $Z_g$  is quasimodular for  $g \geq 2$ . The formula also exhibits a way to concretely compute the number of disconnected covers of given genus and degree.

Recall that the irreducible characters of  $S_d$  are parametrized by Young diagrams of size  $d$ . For example, ...

**Definition 8.1.** Define the *positive half integers*  $\mathbb{Z}_{\geq 0} + \frac{1}{2}$  by

$$\mathbb{Z}_{\geq 0} + \frac{1}{2} = \left\{ \frac{2k+1}{2}; k \in \{0, 1, 2, \dots\} \right\}$$

**Proposition 8.2.** *There is a bijection between the set of Young diagrams of size  $d$  and the set of pairs  $(U, V)$  of finite subsets of  $\mathbb{Z}_{\geq 0} + \frac{1}{2}$  such that  $|U| = |V|$  and  $d = \sum_{u \in U} u + \sum_{v \in V} v$ .*

*Proof.* Consider any a Young diagram of size  $d$ . Starting with the upper left corner, cut it diagonally in two pieces. This gives  $s$  “cut” columns in the lower piece and  $s$  “cut” rows in the upper piece. Let  $u_i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$  denote the number of squares in the  $i$ -th cut row and  $v_i$  the number of squares in the  $i$ -th cut column. Define  $U = \{u_1, \dots, u_s\}$  and  $V = \{v_1, \dots, v_s\}$ . Then  $|U| = |V|$  and  $d = \sum_{u \in U} u + \sum_{v \in V} v$ . Conversely, let two such  $U$  and  $V$  be given. The associated Young diagram is obtained by arranging both  $U$  and  $V$  in ascending order and then iteratively gluing the rows with  $u_i$  squares to the columns with  $v_i$  squares, for the appropriate elements  $u_i \in U$  and  $v_i \in V$  respectively.  $\square$

**Proposition 8.3.** *Let  $\chi$  be the character associated to the Young diagram corresponding to the subsets  $U, V \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$  of equal cardinality  $s$ . Then*

$$\frac{\binom{d}{2} \chi(t)}{\dim(\chi)} = \frac{1}{2} \left( \sum_{i=1}^s u_i^2 - \sum_{i=1}^s v_i^2 \right).$$

*Proof.* See [FH], p. 52.  $\square$

**Definition 8.4.** Define the Laurent series  $\theta(\zeta, q, \lambda)$  in  $\zeta$  with coefficients formal power series in  $q$  and  $\lambda$  as follows:

$$\theta(\zeta, q, \lambda) = \prod_{u \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left( 1 + \zeta q^u e^{u^2 \lambda / 2} \right) \prod_{v \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left( 1 + \zeta^{-1} q^v e^{-v^2 \lambda / 2} \right).$$

**Lemma 8.5.** *The counting function  $\widehat{Z}(q, \lambda)$  is the coefficient of  $\zeta^0$  in the series  $\theta(\zeta, q, \lambda) - 1$ .*

*Proof.* By expanding the product, one finds that  $\theta(\zeta, q, \lambda) = \sum_{U, V \subset \mathbb{Z}_{\geq 0} + \frac{1}{2}} a_{U, V}$ , where

$$a_{U, V} = \zeta^k q^d \exp(\mu_{U, V} \lambda).$$

Here,

1.  $k = |U| - |V|$
2.  $d = \sum_{u \in U} u + \sum_{v \in V} v$
3.  $\mu_{U, V} = \frac{1}{2} \left( \sum_{i=1}^s u_i^2 - \sum_{i=1}^s v_i^2 \right)$ .

Using the bijection in proposition 8.2, let the eigenvalues of the matrix  $M_d$  be indicized by pairs  $(U, V)$  of subsets of  $\mathbb{Z}_{\geq 0} + \frac{1}{2}$  such that  $|U| = |V|$  and  $d = \sum_{u \in U} u + \sum_{v \in V} v$ . By lemma 7.11 and proposition 8.3, the eigenvalue indicized by the pair  $(U, V)$  is equal to  $\mu_{U, V}$ .

Now consider the coefficient of  $\zeta^0$  in  $\theta(\zeta, q, \lambda) - 1$ . There, the coefficient of  $q^d$  is  $\sum_{U, V} \exp \mu_{U, V} \lambda$ , where the  $\mu_{U, V}$  are the eigenvalues of  $M_d$ . By 6.7, this sum is equal to the coefficient of  $q^d$  in  $\widehat{Z}(q, \lambda)$ . This proves the lemma.  $\square$

## 9. Quasimodularity of the generating function

In this section, we use the theorem of Kaneko and Zagier about the generalized Jacobi function found in [KZ] to prove that the generating function  $F_g$  counting connected covers of genus  $g$  is quasimodular for  $g \geq 2$ . Recall that  $F_g$  was defined as the series

$$Z(q, \lambda) = \sum_{g \geq 1} \frac{F_g(q)}{(2g-2)!} \lambda^{2g-2}.$$

For an element  $\tau$  of the upper half plane, set  $q(\tau) = \exp(2\pi i\tau)$ . For convenience, we sometimes write  $q$  instead of  $q(\tau)$ . Also, sometimes  $q$  will be viewed as a formal variable.

**Proposition 9.1.** *Let  $a(x) = \sum_{k \geq 1} a_k x^k$  be a formal power series in  $x$ , with holomorphic functions  $a_k$  on the upper half plane as coefficients. Let  $\exp(a(x)) = \sum_{k \geq 1} b_k x^k$  be its formal exponential. Assume that each of the coefficients  $b_k$  is quasimodular of weight  $kr$ , for some  $r$ . Then the  $a_k$  are also quasimodular of weight  $kr$ .*

*Proof.* This follows essentially by computing by hand the coefficients  $b_i$ . □

**Definition 9.2.** Define the Laurent series  $\Theta(\zeta, q, \lambda)$  in  $\zeta$  with coefficients formal power series in  $q$  and  $\lambda$  as follows:

$$\Theta(\zeta, q, \lambda) = \left( \prod_{n \geq 1} (1 - q^n) \right) \theta(\zeta, q, \lambda).$$

Further, let  $\Theta_0(q, \lambda)$  denote the coefficient of  $\zeta^0$  in  $\Theta(\zeta, q, \lambda)$ .

The following theorem about the quasimodularity of the coefficients of  $\Theta_0$  is proved in [KZ].

**Theorem 9.3** (Kaneko, Zagier). *Let  $\Theta_0(q, \lambda) = \sum_k A_k(q) \lambda^k$  be the constant  $\zeta$ -coefficient of  $\Theta$ . Then the coefficient  $A_k(q)$  is a quasimodular form of weight  $3k$ .*

We may now prove the main result:

**Theorem 9.4** (Dijgraaf). *For  $g \geq 2$ , the function  $F_g \circ q$  is a quasimodular form of weight  $6g - 6$ .*

*Proof.* Lemma 8.5 gives the equality

$$\Theta_0(q, \lambda) = \left( \prod_{n \geq 1} (1 - q^n) \right) (\widehat{Z}(q, \lambda) + 1).$$

By the previous theorem, the coefficient of  $\lambda^{2g-2}$  in this product is quasimodular of weight  $6g - 6$ . By Lemma 4.11 one obtains, after taking the logarithm of both sides of the above equality,

$$\log \Theta_0(q, \lambda) = \sum_{n \geq 1} \log(1 + q^n) + Z(q, \lambda).$$

As seen in section 4.,  $F_1 = -\sum_{n \geq 1} \log(1 + q^n)$ . Hence, in  $\log \Theta_0(q, \lambda)$  the coefficient of  $\lambda^0$  is zero. Thus, we may apply proposition 9.1 and the previous theorem to find that the coefficient of  $\lambda^{2g-2}$  in  $\log \Theta_0(q, \lambda)$ , that is  $F_g(q)/(2g - 2)!$ , is a quasimodular form of weight  $6g - 6$ . This concludes the proof.  $\square$



## 10. Appendix: Calculations

### 10.1. Quasimodular forms

**Calculation 10.1.** This calculation follows the one found in [BO]. Let  $F(\tau) = \sum_{m=1}^M f_m(\tau)Y^{-m}$  be an almost holomorphic modular form,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$ , and  $\tau \in \mathcal{H}$ . Write  $j = c\tau + d$ , and  $a = 6cj/2\pi i$ . Then  $Y^{-1}(\gamma\tau) = a + j^2Y(\tau)^{-1}$ . Hence,

$$\begin{aligned} F(\gamma\tau) &= \sum_{m=1}^M f_m(\gamma\tau)(a + j^2Y^{-1})^m \\ &= \sum_{m=1}^M \sum_{l=0}^m \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l} Y^{-l} \\ &= \sum_{m=1}^M f_m(\gamma\tau) a^m + \sum_{l=1}^M \sum_{m=l}^M \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l} Y^{-l}. \end{aligned}$$

On the other hand,

$$F(\gamma\tau) = \sum_{l=1}^M f_l(\tau) j^k Y^{-l},$$

by the modularity condition. By comparing the coefficients of  $Y^{-l}$ , one obtains the equalities

$$\sum_{m=1}^M f_m(\gamma\tau) a^m = 0 \tag{1}$$

and

$$j^k f_l(\tau) = \sum_{m=l}^M \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l}.$$

Rewriting the second equality yields

$$f_l(\gamma\tau) = f_l(\tau) j^{k-2l} - \sum_{m=l+1}^M \binom{m}{l} f_m(\gamma\tau) a^{m-l}. \tag{2}$$

The latter may be solved recursively, starting by  $f_M$ , to get equalities of the form

$$f_l(\gamma\tau) = (\text{a polynomial in the } f_{\geq l}(\tau), j \text{ and } c). \tag{3}$$

The first two equalities are

$$\begin{aligned} f_M(\gamma\tau) &= f_M(\tau) j^{k-2M} \\ f_{M-1}(\gamma\tau) &= f_{M-1}(\tau) j^{k-2M+2} - \text{const} \cdot f_M(\tau) j^{k-2M+1} c. \end{aligned}$$

In general, a straightforward inductive argument shows that in the summands of the expression (2) for  $f_l(\gamma\tau)$ , the variable  $j$  appears with a power lower than

or equal to  $k - 2l$ . Now let  $r$  be the greatest index such that  $f_r \neq 0$ . Equation (1) finally gives, after substituting back the expressions for  $j$  and  $a$  and using (2) for  $l = r$ , the relation

$$\begin{aligned} 0 &= \kappa_1 f_r(\gamma\tau)(c\tau + d)^r c^r + \sum_{l=r+1}^M \kappa_3 f_l(\gamma\tau)(c\tau + d)^l c^l \\ &= \kappa_1 f_r(\tau)(c\tau + d)^{k-r} c^r - \\ &\quad - \sum_{m=r+1}^M \kappa_2 \binom{m}{r} f_m(\gamma\tau)(c\tau + d)^{m-r} c^{m-r} + \sum_{l=r+1}^M \kappa_3 f_l(\gamma\tau)(c\tau + d)^l c^l, \end{aligned}$$

where the  $\kappa_i$  are some nonzero constants. To obtain a contradiction, choose a point  $\tau$  in the upper half-plane and consider the last relation as a polynomial equation in  $c$  and  $d$ , letting  $P(c, d)$  denote the right-hand side of the equation. First look for the possible coefficients of monomials of the form  $c^r d^{\geq 1}$ . This excludes the third summand from the picture, since there  $c$  will always appear with a power greater than  $r$ . Next look for the possible coefficients of the monomial  $c^r d^{k-r}$ . As seen when recursively solving the equations for  $f_l(\gamma\tau)$ , the second summand will include only terms where  $(c\tau + d)$  appears with a power lower than  $k - r$ . Hence the coefficient of  $c^r d^{k-r}$  in  $P(c, d)$  is  $\kappa_1 f_r(\tau)$ .

Now, if  $c \in \mathbb{Z}$ , then there are infinitely many  $d \in \mathbb{Z}$  such that  $P(c, d) = 0$ . Indeed, there are infinitely many  $d$  with  $\gcd(c, d) = 1$ . For these  $d$ , find  $a, b \in \mathbb{Z}$  such that  $ad - bc = 1$ . Since  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , it follows that  $P(c, d) = 0$ . Similarly, for all  $d \in \mathbb{Z}$ , there are infinitely many  $c$  such that  $P(c, d) = 0$ . It thus follows that  $P(c, d) = 0$  holds for all  $c, d \in \mathbb{C}$ . These remarks may be summarized by the statement that the set of all  $c, d$  belonging to the lower row of some matrix in  $\mathrm{SL}_2(\mathbb{Z})$  is Zariski-dense in  $\mathbb{C}^2$ .

Concluding, since  $P$  is zero as a function on  $\mathbb{C}^2$ , it is also zero as a polynomial, hence the coefficient  $\kappa_1 f_r(\tau)$  is zero. Since  $\tau$  was arbitrary, one finds  $f_r = 0$ , a contradiction.

## References

- [BO] S. Bloch and A. Okounkov. The character of the infinite wedge representation.
- [Dij] R. Dijkgraaf. Mirror symmetry and elliptic curves.
- [DS06] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228. Springer Science & Business Media, 2006.
- [FH] W. Fulton and J. Harris. *Representation Theory: a First Course*.
- [KZ] M. Kaneko and D. Zagier. A generalized Jacobi theta function and quasi-modular forms.
- [Lam] K. Lamotke. *Riemannsche Flächen*.
- [Sera] J.-P. Serre. *A Course in Arithmetic*.
- [Serb] J.-P. Serre. *Linear Representations of Finite Groups*.
- [Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves. 2nd ed.* New York, NY: Springer, 2nd ed. edition, 2009.