Loop Objects in Pointed Derivators

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Derivatore sind abstrakte Mittel, mit denen man Homotopietheorie betreiben kann. Insbesondere können viele Aussagen aus der (klassischen) Homotopietheorie für gewisse Arten von Derivatoren formuliert und bewiesen werden. In dieser Arbeit geht es um eine solche Aussage, nämlich eine Derivatorversion der Tatsache, daß Schleifenräume in der Homotopiekategorie der topologischen Räume eine kanonische Gruppenstruktur besitzen.

Contents

0 Introduction

Motivation

Derivators provide an abstract framework for homotopy theory. In particular, many statements from (classical) homotopy theory can be formulated and proven for certain kinds of derivators. This thesis is about one such statement, namely a "derivator version" of the fact that the loop spaces have a canonical group object structure in the homotopy category of topological spaces.

About This Thesis

The thesis is diveded into three sections. In the first section I try to give a clear and precise description of preadditive and additive categories. It is not much more than an elaboration of the Subsection 2.1 of [2]. In the second section I delve into the main topic of this thesis and show that loop objects in values of a stable derivator are group objects. The main reference for this section is [1]. The third section covers the case of double loop objects and depicts an Eckmann-Hilton argument showing that double loop objects in values of a stable derivator are indeed abelian group objects.

I omitted a general introduction to the theory of derivators, partially because this thesis would be much longer if it introduced every non-trival concept or statement it used and also because there are a few rather elementary introductory texts about this topic (e. g. [2]) which are more detailed than what I could write for this thesis. However, it would be convenient for the reader to get familiar with derivators before reading this thesis.

Even though the introduction is written from a first person perspective, the "mathematical we" will accompany the reader in the main part of the thesis.

Acknowledgments and Thanks

1 Additive Categories and Group Objects

A motivation for showing that loop objects in stable derivators are group objects is that this statement can be used to show that the values of a stable derivator have the structure of an additive category. (In fact, values of a large class of stable derivators have even the structure of a triangulated category and additivity is an important step towards this statement.) In this section, we discuss some concepts and methods which lead to this proof of additivity of values of stable derivators.

Definition 1.1. *A* preadditive category *is a category $\mathscr{A}$ s. t.*

(i) $\mathscr{A}$ is pointed, i. e. has a zero object, which is an object 0 which is both initial and terminal,

(ii) binary (and hence all finite) products and coproducts exist in $\mathscr{A}$,

(iii) for any $X, Y \in \operatorname{ob}\mathscr{A}$, the morphism

$$(\mathrm{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \mathrm{id}_Y) \colon X \amalg Y \to X \times Y$$

is an isomorphism, where $0_{X,Y} \colon X \to 0 \to Y$ resp. $0_{Y,X} \colon Y \to 0 \to X$ is the unique morphism which factors through a zero object.

Notation 1.2.

- *Biproducts in the above sense will be denoted by $\_ \oplus \_$.*
- *If X, Y, X' resp. Y' are objects of a preadditive category and $f_{X,X'} \colon X \to X'$, $f_{Y,X'} \colon Y \to X'$, $f_{X,Y'} \colon X \to Y'$ resp. $f_{Y,Y'} \colon Y \to Y'$ are some morphisms, then we denote the morphism*

  $$(f_{X,X'} \times f_{X,Y'}) \amalg (f_{Y,X'} \times f_{X,X'}) \colon X \oplus Y \to X' \oplus Y'$$

  by

  $$\begin{pmatrix} f_{X,X'} & f_{Y,X'} \\ f_{X,Y'} & f_{Y,Y'} \end{pmatrix}.$$

  Note that, using the universal properties of products and coproducts, any morphism $f \colon X \oplus Y \to X' \oplus Y'$ can be written as

  $$f = \begin{pmatrix} \mathrm{pr}_{X'} \circ f \circ \mathrm{in}_X & \mathrm{pr}_{X'} \circ f \circ \mathrm{in}_Y \\ \mathrm{pr}_{Y'} \circ f \circ \mathrm{in}_X & \mathrm{pr}_{Y'} \circ f \circ \mathrm{in}_Y \end{pmatrix}.$$

  Matrices of different sizes are constructed similarly.
- *We will also use common abuses of notation such as denoting an identity morphism by 1 or a morphism that factors through a zero object by 0.*

Remark 1.3. *A preadditive category is enriched over the category* **AbMon** *of abelian monoids. Indeed, for any $X, Y \in \operatorname{ob}\mathscr{A}$, setting*

$$f + g\colon X \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix}f & 0\\ 0 & g\end{pmatrix}} Y \oplus Y \xrightarrow{\begin{pmatrix}1\\ 1\end{pmatrix}} Y$$

for $f, g \in \operatorname{Hom}_{\mathscr{A}}(X, Y)$ yields an abelian monoid structure on $\operatorname{Hom}_{\mathscr{A}}(X, Y)$ with neutral element $0_{X,Y}$ and for any $X, Y, Z \in \operatorname{ob}\mathscr{A}$, the composition map

$$\_ \circ \_ \colon \operatorname{Hom}_{\mathscr{A}}(Y, Z) \times \operatorname{Hom}_{\mathscr{A}}(X, Y) \to \operatorname{Hom}_{\mathscr{A}}(X, Z)$$

is bilinear w. r. t. this "addition operation".

Furthermore, a straightforward computation shows that composing morphisms corresponds to multiplying their matrix representations.

Remark 1.4. *The previous remark implies in particular that any object X of a preadditive category has the structure of an abelian monoid object given by the codiagonal morphism $\nabla := \begin{pmatrix}1 & 1\end{pmatrix} : X \oplus X \to X$ and the "unit" $0 \to X$. Dually, X has also the structure of a coabelian comonoid object given by the diagonal morphism $\Delta := \begin{pmatrix}1\\ 1\end{pmatrix} : X \oplus X \to X$ and the "counit" $X \to 0$.*

As the prefix "pre" suggests, preadditive categories are not quite what we are looking for. We will get to the concept of additive categories by requiring that additive inverses of morphisms exist:

Proposition 1.5. *The following are equivalent for a preadditive category $\mathscr{A}$:*

(i) For any $X \in \operatorname{ob}\mathscr{A}$ the "shear morphism"

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} : X \oplus X \to X \oplus X$$

is an isomorphism.

(ii) For any $X \in \operatorname{ob}\mathscr{A}$ the identity morphism id_X has an additive inverse in $\operatorname{End}_{\mathscr{A}}(A)$.

(iii) For any $X, Y \in \operatorname{ob}\mathscr{A}$, each $f \in \operatorname{Hom}_{\mathscr{A}}(X, Y)$ has an additive inverse.

(iv) For any $X \in \operatorname{ob}\mathscr{A}$, the abelian monoid object $(X, \nabla, 0 \to X)$ is an (abelian) group object.

(v) For any $X \in \operatorname{ob}\mathscr{A}$, the coabelian comonoid object $(X, \Delta, X \to 0)$ is a (coabelian) cogroup object.

Proof. "(*i*) ⇒ (*ii*)": Let the inverse of the shear morphism of X be given by

$$\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} : X \oplus X \to X \oplus X.$$

Then we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} = \begin{pmatrix} j_{1,1} + j_{2,1} & j_{1,2} + j_{2,2} \\ j_{2,1} & j_{2,2}. \end{pmatrix}$$

Hence $j_{1,2} = 0_{X,X}$ and $j_{1,1} = j_{2,2} = \mathrm{id}_X$. This yields

$$\mathrm{id}_X + j_{1,2} = j_{2,2} + j_{1,2} = 0_{X,X},$$

so $j_{1,2}$ is an additive inverse of id_X.

"(*ii*) ⇒ (*iii*)": Let $-\mathrm{id}_X$ be an additive inverse for id_X. Then the bilinearity of composition yields

$$f + f \circ (-\mathrm{id}_X) = f \circ \mathrm{id}_X + f \circ (-\mathrm{id}_X) = f \circ (\mathrm{id}_X + (-\mathrm{id}_X)) = f \circ 0_{X,X} = 0_{X,Y},$$

i. e. $f \circ (-\mathrm{id}_X)$ is an additive inverse for f.

"(*iii*) ⇒ (*iv*)": Note that X is a group object in $\mathscr{A}$ iff its represented functor $\mathrm{Hom}_{\mathscr{A}}(\_, X)$ factors through the category **Grp** of groups. Since X is an abelian monoid object, we already know that $\mathrm{Hom}_{\mathscr{A}}(\_, X)$ factors through **AbMon**. Now the fact that for each $Y \in \mathrm{ob}\,\mathscr{A}$ each $f \in \mathrm{Hom}_{\mathscr{A}}(Y, X)$ has an additive inverse implies that the abelian monoids $(\mathrm{Hom}_{\mathscr{A}}(Y, X), +_{Y,X}, 0_{Y,X})$ are in fact abelian groups. Since all monoid homomorphisms between groups are already homomorphisms of groups, this means that $\mathrm{Hom}_{\mathscr{A}}(\_, X)$ factors through the category of (abelian) groups.

"(*iv*) ⇒ (*v*)": If X is a group object with the "multiplication" given by ∇, there exists a morphism $j : X \to X$ s. t.

$$0_{X,X} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} = \mathrm{id}_X \circ \mathrm{id}_X + \mathrm{id}_X \circ j = \mathrm{id}_X + j.$$

Hence for comonoid structure on X we obtain

$$\begin{pmatrix} 1 & j \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathrm{id}_X \circ \mathrm{id}_X + j \circ \mathrm{id}_X = \mathrm{id}_X + j = 0_{X,X}.$$

A similar argument shows that the fact that j is also a "left inverse for the multiplication of X" implies that j is also a "left inverse for the comultiplication of X". In total, we obtain that $(X, \Delta, X \to 0, j)$ is a cogroup object.

"(*v*) ⇒ (*i*)": Let $j : X \to X$ be the "coinverse" morphism w. r. t. Δ. Then calculations similar to the ones in the proof of the previous implication yield that $\mathrm{id}_X + j = 0_{X,X} = j + \mathrm{id}_X$. Hence we obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & j+1 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and
$$\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1+0 & 1+j\\ 0+0 & 0+1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$
Since $\mathrm{id}_{X\oplus X} = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, we see that the shear morphism is an isomorphism. $\square$

Definition 1.6. *A preadditive category is called* additive *if it satisfies one (and hence all) of the conditions in the previous proposition.*

Remark 1.7. *Additive categories are enriched over* **Ab**.

2 Loop Objects

In this section, we prove our main result by first showing that loop objects in values of pointed derivators are monoid objects and then showing that there are inversion morphisms for the multiplication morphisms of loop objects.

For the rest of this section let $\mathscr{D}$ be a pointed derivator.

2.1 Loop Objects as Simplicial Objects

As a preparation for the monoid structure, we will now show that loop objects fulfill slightly more general conditions than that for a simplicial object.

Notation 2.1.

- *Let* $\langle n\rangle := \{0,\dots,n\}$ *for* $n \in \mathbb{N}$. *We will consider these as objects of the category* **Fin** *of finite sets or equivalently finite discrete categories.*
- *Let* $\_^{\triangleright}\colon \mathbf{Fin} \to \mathbf{Cat}$ *be the cone functor, i. e. the functor which adds a terminal object* ∞ *to a given category. Let* $\lrcorner_n := \langle n\rangle^{\triangleright}$.

Definition 2.2. *For* $a \in \mathbf{Fin}$ *let*

$$\omega_a\colon \mathscr{D}(\ast) \xrightarrow{\infty_!} \mathscr{D}(a^{\triangleright}) \xrightarrow{\lim_{a^{\triangleright}}} \mathscr{D}(\ast).$$

For $n \in \mathbb{N}$ *we will abuse notation and write* ω_n *for* $\omega_{\langle n\rangle}$. *In particular, we have* $\Omega \cong \omega_1 = \lim_{\lrcorner_1} \circ \infty_!\colon \mathscr{D}(\ast) \to \mathscr{D}(\lrcorner_1) \to \mathscr{D}(\ast)$ *for the loop functor.*

Lemma 2.3. *The assignment* $a \mapsto \omega_a$ *can be made into a functor*

$$\omega\colon \mathbf{Fin}^{\mathrm{op}} \to \mathrm{END}_{\mathbf{CAT}}(\mathscr{D}(\ast)).$$

Proof. For $a \in \operatorname{ob}\mathbf{Fin}$, ω_a is an endofunctor of $\mathscr{D}(\ast)$ by construction.

For functoriality, we consider $a, b \in \operatorname{ob}\mathbf{Fin}$ and $f\colon a \to b$. Then we have two diagrams

$$\begin{tikzcd}
\ast \arrow[r] \arrow[d, "\infty"'] & \ast \arrow[d, "\infty"] \\
a^{\triangleright} \arrow[r, "f^{\triangleright}"] \arrow[d] & b^{\triangleright} \arrow[d] \\
\ast \arrow[r] & \ast
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
\mathscr{D}(\ast) \arrow[d, "\infty_!"'] \arrow[dr, phantom, "\Searrow"] & \mathscr{D}(\ast) \arrow[l] \arrow[d, "\infty_!"] \\
\mathscr{D}(a^{\triangleright}) \arrow[d, "\lim_{a^{\triangleright}}"'] \arrow[dr, phantom, "\Nwarrow"] & \mathscr{D}(b^{\triangleright}) \arrow[l, "(f^{\triangleright})^{*}"] \arrow[d, "\lim_{b^{\triangleright}}"] \\
\mathscr{D}(\ast) & \mathscr{D}(\ast) \arrow[l]
\end{tikzcd}
, \tag{1}$$

where the second one is obtained from the first by applying $\mathscr{D}$ and then using the appropriate mates.

Now we want to show that the upper natural transformation on the right is an isomorphism and then define the natural transformation $\omega_f \colon \omega_b \Rightarrow \omega_a$ as the pasting of the two squares on the right.

Note that we can detect such isomorphisms pointwise. In order to do that, we consider an $x \in \mathrm{ob}(a)$ which yields a diagram

$$\begin{tikzcd}
(\infty/x) \arrow[r, "\pi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swarrow"] & \ast \arrow[r] \arrow[d, "\infty"] \arrow[dr, phantom, "\swarrow"] & \ast \arrow[d, "\infty"] \\
\ast \arrow[r, "x"'] & a^{\triangleright} \arrow[r, "f^{\triangleright}"'] & b^{\triangleright}
\end{tikzcd} .$$

Then we know that the mate transformation $\pi_! \pi^* \Rightarrow x^* \infty_!$ is an isomorphism since the square on the left is a slice square and hence homotopy exact. Futhermore, we have

$$(\infty/x) \cong \begin{cases} \varnothing & x \neq \infty \\ \ast & x = \infty \end{cases} .$$

Since $f^{\triangleright}(x) = \infty$ iff $x = \infty$, this yields that the pasting of the two squares is a also slice square, hence homotopy exact, which means that the mate transformation $\pi_! \pi^* \Rightarrow (f^{\triangleright}(x))^* \infty_!$ is an isomorphism. Hence, in total, we obtain that the mate transformation $x^* \infty_! \Rightarrow (f^{\triangleright}(x))^* \infty_!$ is an isomorphism.

We can now define $\omega_f \colon \omega_b \Rightarrow \omega_b$ to be the pasting of the inverse of $\infty_! \Rightarrow (f^{\triangleright})^* \infty_!$ with $\lim_{b^{\triangleright}} \Rightarrow \lim_{a^{\triangleright}} (f^{\triangleright})^*$. This construction is compatible with composition of maps since mates are compatible with pastings. Furthermore, identities are mapped to identites since all the natural transformations in (1) are identities if f is an identity map. $\square$

Corollary 2.4. *For $X \in \mathrm{ob}(\mathscr{D}(\ast))$, $(\omega_n X)_{n \in \mathbb{N}}$ can be viewed as a simplicial object since the simplex category* $\mathbf{\Delta}$ *is a subcategory of* **Fin** *(which is not full).*

2.2 Loop Objects as Monoid Objects

The next step in this section is showing that the simplicial objects associated with loop objects are trivial in the zeroth level and satisfy the Segal condition, which means that loop objects are monoid objects.

Remark 2.5. *For $X \in \mathrm{ob}(\mathscr{D}(\ast))$ we have $\omega_0 X \cong 0^* \infty_! X$ since 0 is the terminal object of $\langle 0 \rangle$, and hence $\omega_0 \cong 0$ since $\infty \colon \ast \to \langle 0 \rangle$ is a cosieve.*

Proposition 2.6. *Let $n > 1$. We define $i_n \colon \langle n-1 \rangle \to \langle n \rangle$ to be the inclusion and $i'_n \colon \langle 1 \rangle \to \langle n \rangle$ to be the function with $i'_n(0) = n-1$ resp. $i'_n(1) = n$.*

Then the natural transformation $\alpha_n \colon \omega_n \Rightarrow \omega_{n-1} \times \omega_1$ induced by the functor $k_n := i_n^{\triangleright} \amalg {i'_n}^{\triangleright} \colon \lrcorner_{n-1} \amalg \lrcorner_1 \to \lrcorner_n$ is an isomorphism.

Proof. Let J_n be the category which is obtained from $\lrcorner_n$ by adding two objects w_0, w_1 with morphisms $w_0 \to k$ for $0 \leqslant k \leqslant n-1$ resp. $w_1 \to k$ for $n-1 \leqslant k \leqslant n$ (and resulting compositions), and let $j_n \colon \lrcorner_n \to J_n$ denote its inclusion functor.

Let $\lrcorner$ be the full subcategory of J_n containing w_0, w_1 and $n-1$ (which is isomorphic to $\lrcorner_1$), and let l_n denote its inclusion functor. Since $n-1$ is terminal in $\lrcorner$, we will denote it also by ∞. Note that l_n has a right adjoint r_n given by

$$r_n(x) = \begin{cases} w_0 & x \in \{w_0, 0, \ldots, n-2\} \\ w_1 & x \in \{w_1, n\} \\ n-1 & x \in \{n-1, \infty\} \end{cases} .$$

for $x \in \operatorname{ob} J_n$, which defines the images of morphisms uniquely. Hence we have $l_n^* \cong (r_n)_*$.

Then, using the natural equivalence $\mathscr{D}(A \amalg B) \simeq \mathscr{D}(A) \times \mathscr{D}(B)$ for $A, B \in \operatorname{ob} \mathbf{Cat}$ and appropriate mates, we obtain a diagram

$$\begin{tikzcd}
\mathscr{D}(\ast) \arrow[r, "(\mathrm{id} \amalg \mathrm{id})^*"] \arrow[d, "(\infty_n)_!"'] \arrow[dr, phantom, "\Swarrow"] & \mathscr{D}(\ast \amalg \ast) \arrow[d, "(\infty_{n-1} \amalg \infty_1)_!"] & \\
\mathscr{D}(\lrcorner_n) \arrow[r, "k_n^*"] \arrow[d, "(j_n)_*"'] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}(\lrcorner_{n-1} \amalg \lrcorner_1) \arrow[d, "(\pi_{n-1} \amalg \pi_1)_*"] & \\
\mathscr{D}(J_n) \arrow[r, "(w_0 \amalg w_1)^*"] \arrow[d, "l_n^* \cong (r_n)_*"'] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}(\ast \amalg \ast) \arrow[r, "(\pi_{\ast \amalg \ast})_*"] \arrow[d, "(w_0 \amalg w_1)_*"] \arrow[dr, phantom, "\cong"] & \mathscr{D}(\ast) \arrow[d, "\mathrm{id}_*"] \\
\mathscr{D}(\lrcorner) \arrow[r, "\mathrm{id}^*"] & \mathscr{D}(\lrcorner) \arrow[r, "(\pi_1)_*"] & \mathscr{D}(\ast)
\end{tikzcd} .$$

Under the equivalences mentioned above the upper natural transformation is given by

$$(\infty_{n-1})_! \times (\infty_1)_! \Rightarrow \left((i_n^{\triangleright})^* \times (i_n'^{\,\triangleright})^* \right) (\infty_n)_!,$$

which is the product of the natural transformations which occur in the definition of ω_{i_n} resp. $\omega_{i'_n}$ (see (1) in the proof of Lemma 2.1). Hence it is an isomorphism as the product of two natural isomorphisms. Furthermore, the right square in the last row commutes up to isomorphism since $\pi_1(w_0 \amalg w_1) = \pi_{\ast \amalg \ast}$.

All in all, the diagram above yields a natural transformation from

$$(\pi_1)_*(r_n)_*(j_n)_*(\infty_n)_! \cong (\pi_n)_*(\infty_n)_! \cong \omega_n$$

to

$$(\pi_{\ast \amalg \ast})_*(\pi_{n-1} \amalg \pi_1)_*(\infty_{n-1} \amalg \infty_1)_! \cong ((\pi_{n-1})_* \infty_{n-1}) \times ((\pi_1)_* \infty_1) \cong \omega_{n-1} \times \omega_1$$

which is the α_n mentioned in the statement of this proposition. We now want to show that (certain restrinctions of) the natural transformations in the remaining two squares are isomorphisms, which will imply that α_n is an isomorphism.

For the middle square we consider diagrams of the form

$$\begin{tikzcd}
(x/(\pi_{n-1} \amalg \pi_1)) \arrow[r, "p_{x,\pi_{n-1}\amalg\pi_1}"] \arrow[d, "\pi_{(x/(\pi_{n-1}\amalg\pi_1))}"'] \arrow[dr, phantom, "\Nearrow"] & \lrcorner_{n-1} \amalg \lrcorner_1 \arrow[r, "k_n"] \arrow[d, "\pi_{n-1}\amalg\pi_1"'] \arrow[dr, phantom, "\Nearrow"] & \lrcorner_n \arrow[d, "j_n"] \\
\ast \arrow[r, "x"'] & \ast \amalg \ast \arrow[r, "w_0 \amalg w_1"'] & J_n
\end{tikzcd}$$

for $x \in \mathrm{ob}(\ast \amalg \ast) = \{*_0, *_1\}$.

Then we have

$$(*_0/(\pi_{n-1} \amalg \pi_1)) \cong \lrcorner_{n-1} \quad \text{and} \quad (*_1/(\pi_{n-1} \amalg \pi_1)) \cong \lrcorner_1$$

where under this identification $p_{*_0,\pi_{n-1}\amalg\pi_1}$ resp. $p_{*_1,\pi_{n-1}\amalg\pi_1}$ is given by the inclusion ι_0 resp. ι_1 of the corresponding category. Since the left square is a slice square, this means that

$$*_0^*(\pi_{n-1} \amalg \pi_1)_* \Rightarrow (\pi_{(*_0/(\pi_{n-1}\amalg\pi_1))})_*(p_{*_0,\pi_{n-1}\amalg\pi_1})^* \cong (\pi_{n-1})_*\iota_0^*$$

and

$$*_1^*(\pi_{n-1} \amalg \pi_1)_* \Rightarrow (\pi_{(*_1/(\pi_{n-1}\amalg\pi_1))})_*(p_{*_1,\pi_{n-1}\amalg\pi_1})^* \cong (\pi_1)_*\iota_1^*$$

are isomorphisms.

On the other hand, we also have

$$(w_0/j_n) \cong \lrcorner_{n-1} \quad \text{and} \quad (w_1/j_n) \cong \lrcorner_1,$$

where under this identification p_{w_0,j_n} is given by $i_n^{\rhd} = k_n\iota_0$ and p_{w_1,j_n} is given by $i_n'^{\rhd} = k_n\iota_1$. Hence the pasting of the above squares is (up to isomorphisms) also a slice square, so the natural transformations

$$((w_0 \amalg w_1)*_0)^*(j_n)_* = w_0^*(j_n)_* \Rightarrow (\pi_{(w_0/j_n)})_*(p_{w_0,j_n})^* \cong (\pi_{n-1})_*(i_n^{\rhd})^*$$

and

$$((w_0 \amalg w_1)*_1)^*(j_n)_* = w_1^*(j_n)_* \Rightarrow (\pi_{(w_1/j_n)})_*(p_{w_1,j_n})^* \cong (\pi_1)_*(i_n^{\rhd})^*$$

are also isomorphisms.

Combining these isomorphisms, we see that

$$\begin{aligned}
*_0^*(\pi_{n-1} \amalg \pi_1)_* k_n^* &\cong (\pi_{n-1})_*\iota_0^* k_n^* \\
&\cong (\pi_{n-1})_*(i_n^{\rhd})^* \cong w_0^*(j_n)_* \\
&\cong ((w_0 \amalg w_1)*_0)^*(j_n)_* \cong *_0^*(w_0 \amalg w_1)^*(j_n)_*
\end{aligned}$$

and

$$\begin{aligned}
*_1^*(\pi_{n-1}\amalg\pi_1)_*k_n^* &\cong (\pi_{n-1})_*\iota_1^*k_n^* \\
&\cong (\pi_{n-1})_*(i_n'^{\triangleright})^* \cong w_1^*(j_n)_* \\
&\cong ((w_0\amalg w_1)*_1)^*(j_n)_* \cong *_1^*(w_0\amalg w_1)^*(j_n)_*.
\end{aligned}$$

Since mates are compatible with pastings this means that the natural transformation $x^*(w_0\amalg w_1)^*(j_n)_* \Rightarrow x^*(\pi_{n-1}\amalg\pi_1)_*k_n^*$ is an isomorphism for all $x \in \mathrm{ob}(\ast\amalg\ast)$, hence it is an isomorphism as isomorphisms can be detected pointwise.

Note that, in general, the natural transformation

$$l_n^* \cong (r_n)_* \Rightarrow (w_0\amalg w_1)_*(w_0\amalg w_1)^*$$

in the first square of the last row is not an isomorphism for all $X \in \mathscr{D}(J_n)$. We are going to "fix" this by restricting our attention to $\mathrm{essim}((j_n)_*(\infty_n)_!)$.

First, we compute $(n-1)^*(j_n)_*X'$ for $X' \in \mathrm{essim}(\infty_!)$: Consider the slice square

$$\begin{tikzcd}
(n-1/j_n) \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\nearrow"] & \lrcorner_n \arrow[d, "j_n"] \\
\ast \arrow[r, "n-1"'] & J_n
\end{tikzcd}.$$

Then we know that $(n-1)^*(j_n)_* \Rightarrow \pi_*p^*$ is an isomorphism.

Now $(n-1/j_n)$ is isomorphic to the full subcategory K_n of $\lrcorner_n$ spanned by $n-1$ and ∞, where p corresponds to the inclusion $K_n \to \lrcorner_n$ under this identification. Hence we see that $\pi_*p^* \cong (n-1)^*p^* \cong (p(n-1))^*$ since $n-1$ is the initial object of K_n. Therefore $(n-1)^*(j_n)_* \cong (n-1)^*$, where the former $n-1$ is the object in J_n and the latter the one in $\lrcorner_n$. Since $\infty\colon \ast \to \lrcorner_n$ is a cosieve we know that $(n-1)^*X' \cong 0$ for $X' \in \mathrm{essim}(\infty_!)$, so we obtain $(n-1)^*(j_n)_*X' \cong 0$.

This means that for $X \in \mathrm{essim}((j_n)_*(\infty_n)_!)$ we have $\infty^*l_n^*X \cong (l_n\infty)^*X \cong (n-1)^*X \cong 0$. On the other hand, for any $Y \in \mathscr{D}(\ast\amalg\ast)$, we have $\infty^*(w_0\amalg w_1)_*Y \cong 0$ since $w_0\amalg w_1$ is a sieve. Hence $l_n^* \cong (r_n)_*$ and $(w_0\amalg w_1)_*(w_0\amalg w_1)^*$ agree on $\infty = n-1$ for $X \in \mathrm{essim}((j_n)_*(\infty_n)_!)$.

We now consider w_i for $i \in \{0,1\}$. In the slice square

$$\begin{tikzcd}
(w_i/w_0\amalg w_1) \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\nearrow"] & \ast\amalg\ast \arrow[d, "w_0\amalg w_1"] \\
\ast \arrow[r, "w_i"'] & \lrcorner
\end{tikzcd},$$

$(w_i/w_0\amalg w_1)$ can be identified with $\ast$ and p with $*_i\colon \ast \to \ast\amalg\ast$. Hence we see that $w_i^*(w_0\amalg w_1)_* \Rightarrow \pi_*w_i^* \cong w_i^*$ is an isomorphism. This yields

$$\begin{aligned}
w_i^*(w_0\amalg w_1)_*(w_0\amalg w_1)^* &\overset{\cong}{\Rightarrow} w_i^*(w_0\amalg w_1)^* = ((w_0\amalg w_1)w_i)^* \\
&= w_i^* = (l_nw_i)^* = w_i^*l_n^* \cong w_i^*(r_n)_*,
\end{aligned}$$

so $w_i^*(r_n)_* \Rightarrow w_i^*(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ is an isomorphism by the compatibility of mates with pastings.

All in all, $(r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ is an isomorphism pointwise, so it is indeed an isomorphism. This means that the last remaining square is also filled with an isomorphism, so $\alpha_n \colon \omega_n \Rightarrow \omega_{n-1} \times w_1$ is an isomorphism in total. □

Corollary 2.7. *Let $\mu\colon \langle 1\rangle \to \langle 2\rangle$ be the map with $\mu(0) = 0$ resp. $\mu(1) = 2$ and let ε be the unique map from $\langle 1\rangle$ to $\langle 0\rangle$.*

Then for any $X \in \mathrm{ob}(\mathscr{D}(\circledast))$, $\omega_1 X \cong \Omega X$ has a monoid object structure given by the multiplication

$$m_X \colon \omega_1 X \times \omega_1 X \xrightarrow{(\alpha_2^{-1})_X} \omega_2 X \xrightarrow{(\omega_\mu)_X} \omega_1 X$$

and the unit

$$0\colon 0 \xrightarrow{\cong} \omega_0 X \xrightarrow{(\omega_\varepsilon)_X} \omega_1 X.$$

Proof. Remark 2.5 and Proposition 2.6 imply that the Segal morphism

$$\omega_n X \to (\omega_1 X)^{\times_{\omega_0 X} n} \cong (\omega_1 X)^n$$

is an isomorphism for any $n \in \mathbb{N}$, i. e. that $(\omega_n X)_{n\in\mathbb{N}}$ satisfies the Segal condition.

Hence $\omega_1 X$ is a category object, where the composition is given by $\omega_\mu\colon \omega_2 \to \omega_1$ and identity morphisms are given by $\omega_\varepsilon\colon \omega_0 \to \omega_1$. Since $\omega_0 X \cong 0$, this means that $\omega_1 X$ is a monoid object with the given multiplication and unit. □

2.3 Loop Objects as Group Objects

The last step of our consideration is the construction of inverses for the multiplication of loop objects, concluding that loop objects are group objects.

Proposition 2.8. *Let $\sigma\colon \langle 1\rangle \to \langle 1\rangle$ be the only non-trivial automorphism, i. e. the map swapping 0 and 1.*

Then, for any $X \in \mathscr{D}(\circledast)$, there is an inversion morphism for the multiplication of $\Omega X \cong \omega_1 X$ which is given by $(\omega_\sigma)_X \colon \omega_1 X \to \omega_1 X$.

Proof. We have to show that the composition $z := m_X \circ (\mathrm{id}_X \times (w_\sigma)_X)$ is the zero morphism. In order to do that, we will describe z as a morphism which factors through $\omega_2 X$.

Let $\phi\colon \langle 2\rangle \to \langle 1\rangle$ be the map with $\phi(0) = 0 = \phi(2)$ and $\phi(1) = 1$, Then we have a diagram

$$\begin{tikzcd}
\mathscr{D}(\ast) \arrow[rr, "(\mathrm{id}\amalg\mathrm{id})^*"] \arrow[dd, "\infty_!"'] & & \mathscr{D}(\ast \amalg \ast) \arrow[dd, "(\infty\amalg\infty)_!"] \\
& \mathscr{D}(\ast) \arrow[ul, "\mathrm{id}^*"] \arrow[ur, "(\mathrm{id}\amalg\mathrm{id})^*"'] \arrow[dd, "\infty_!"] & \\
\mathscr{D}(\lrcorner_2) \arrow[rr, "k_2^*"] \arrow[dd, "\pi_*"'] & & \mathscr{D}(\lrcorner_1 \amalg \lrcorner_1) \arrow[dd, "\pi_*"] \\
& \mathscr{D}(\lrcorner_1) \arrow[ul, "(\phi^{\triangleright})^*"] \arrow[ur, "(\mathrm{id}\amalg\sigma^{\triangleright})^*"'] \arrow[dd, "\pi_*"] & \\
\mathscr{D}(\ast) \arrow[rr, "\mathrm{id}^*"] & & \mathscr{D}(\ast) \\
& \mathscr{D}(\ast) \arrow[ul, "\mathrm{id}^*"] \arrow[ur, "\mathrm{id}^*"'] &
\end{tikzcd}$$

in which the horizontal triangles commute.

Hence the vertical squares on the right side can be seen as pastings of the vertical squares on the left side and the vertical squares at the back. This means that $\mathrm{id}_X \times (w_\sigma)$ (pasting of the squares on the right) can be identified as $\alpha_2 \circ \omega_\phi$ (pasting of the pastings of the squares on the left resp. at the back).

Using the definition of m_X, we obtain that

$$\begin{aligned}
z &= m_X \circ (\mathrm{id}_X \times (w_\sigma)_X) \\
&= (\omega_\mu)_X \circ (\alpha_2^{-1})_X \circ (\alpha_2)_X \circ (\omega_\phi)_X \\
&= (\omega_\mu)_X \circ (\omega_\phi)_X = (\omega_{\phi\circ\mu})_X.
\end{aligned}$$

Now note that $\phi \circ \mu$ factors through $\langle 0\rangle$ as $\phi(\mu(0)) = \phi(0) = 0 = \phi(2) = \phi(\mu(0))$. Hence $z = (\omega_{\phi\circ\mu})_X$ factors through $\omega_0 X \cong 0$, so it is indeed the zero morphism. $\square$

3 Double Loop Objects

References

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