

# Loop Objects in Pointed Derivators

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## Zusammenfassung

Ein wichtiger Ansatz in der Homotopietheorie (von topologisch Räumen, Kettenkomplexen usw.) ist das Studium von sogenannten *Homotopie(ko)limites* bzw. allgemeiner von *Homotopiekernerweiterungen*. Diese verallgemeinern viele homotopietheoretische Konstruktionen und haben nützliche Eigenschaften, die ähnlich zu den von klassischen Kanerweiterungen sind. *Derivatoren* axiomatisieren diesen Kalkül von (Homotopie-)Kanerweiterungen und erlauben uns somit, mit abstrakten Mitteln Homotopietheorie zu betreiben.

Wir betrachten zum Beispiel den Schleifenraum  $\Omega(X, x)$  eines punktierten topologischen Raumes  $(X, x)$ . Dieser kann als das Homotopiepullback des Diagramms

$$\begin{array}{ccc} & * & \\ & \downarrow & \\ * & \longrightarrow & (X, x) \end{array}$$

aufgefasst werden, wobei  $*$  ein einpunktiger Raum ist. Außerdem besitzt  $\Omega(X, x)$  in der Homotopiekategorie von punktierten Räumen die Struktur eines Gruppenobjekts, was durch Verkettung und Umkehrung von Schleifen gegeben ist.

Es gibt in der Tat analoge Konstruktionen für sogenannte „punktierte Derivatoren“. Man kann auch noch zeigen, dass diese „Schleifenobjekte“ die Struktur eines Gruppenobjekts besitzen, was das Hauptthema dieser Arbeit ist.



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# Introduction

## Motivation

One way of studying homotopy theory (of topological spaces, chain complexes etc.) is considering so-called *homotopy (co)limits* or more generally *homotopy Kan extensions*. Besides encompassing many homotopy theoretical constructions, these have very useful formal properties similar to the properties of classical Kan extensions. The concept of a *derivator* provides an abstract framework for homotopy theory by axiomatizing this calculus of (homotopy) Kan extensions.

For example, consider the loop space  $\Omega(X, x)$  of a given pointed topological space  $(X, x)$ . This can be thought of as the homotopy pullback of the diagram

$$\begin{array}{ccc} & * & \\ & \downarrow & \\ * & \longrightarrow & (X, x) \end{array},$$

where  $*$  is a space with only one point. Moreover, note that in the homotopy category of pointed spaces,  $\Omega(X, x)$  has a group object structure given by concatenation and inversion of loops.

There are indeed analogous constructions in the setting of so-called “pointed derivators”. Furthermore, one can show that these “loop objects” have a canonical group object structure, which is the main topic of this thesis.

## About This Thesis

The thesis starts with a review of derivators. It is by far not a detailed introduction to the theory of derivators, but merely a summary of some results which are needed later.

Having all required concepts at hand, the first section of the main part deals with a construction which yields a group object structure on loop objects. Here some generalities on simplicial objects are needed, which are covered in the first appendix.

From this point on, the technical details about derivators are not very important. One can, for instance, show that twofold loop objects have an abelian group object structure by using a formal Eckmann–Hilton argument. This is done in the second section of the main part.

The main part ends with a short section on applications. The main application is the additivity of so-called “stable derivators”. For this some general statements about additive categories are needed, which are dealt with in the second appendix. Furthermore, it is also mentioned that these results can be applied to not only the underlying category of a derivator,

but to all of its values.

Even though the introduction is written from a first person perspective, the “mathematical we” will accompany the reader in the main part of the thesis.

## **Acknowledgments and Thanks**



## 0 Preliminaries

### Conventions and Notations

When we consider a *category*, it is implicit that the category in consideration is *locally small*, i. e. the homomorphism classes are sets. In a few situations we will make this explicit in order to emphasize that the homomorphism classes are indeed sets.

There are two cases, namely Definition 0.6 and Lemma 1.3, where we talk about “functors” into “categories” which are not necessarily locally small. This is also emphasized by the usage of capital letters in notations like **CAT** and **END**. In these cases, one must indeed confront some “size issues” which we do not want to deal with in detail. Our approach is considering these statements as “metatheoretical” ones and explicitly writing down what they mean.

Now we introduce some concepts and notations which will be used throughout the thesis.

**Notation 0.1.** Let  $\ast$  denote the category which has a unique object  $\ast$  with  $\text{End}_{\ast}(\ast) = \text{id}_{\ast}$ , which is a terminal object in the category **Cat** of small categories.

**Notation 0.2.** Let  $\mathcal{C}$  be a category and  $A$  a small category. Then  $\mathcal{C}^A$  will denote the category of functors  $A \rightarrow \mathcal{C}$  with natural transformations between such functors as morphisms.

Note that in this case  $\mathcal{C}^A$  is indeed a locally small category since the class of natural transformations between two functors  $F, G: A \rightarrow \mathcal{C}$  can essentially be seen as a subclass of  $\text{Hom}_{\mathbf{Set}}(\text{ob } A, \prod_{a \in \text{ob } A} \text{Hom}_{\mathcal{C}}(F(a), G(a)))$  which is a set.

**Notation 0.3.** Let  $\mathcal{C}$  be a category which has products and coproducts.

For a family  $(X_i)_{i \in I}$  of objects in  $\mathcal{C}$  we will denote the structure morphisms of the product by  $\text{pr}_i: \prod_{i \in I} X_i \rightarrow X_i$ . Given a family of morphisms  $(f_i: Y \rightarrow X_i)_{i \in I}$  in  $\mathcal{C}$ , we will denote the induced morphism to the product by  $\prod_{i \in I} f_i: Y \rightarrow \prod_{i \in I} X_i$ . If  $I = \{1, \dots, n\}$  is finite, the product will alternatively be denoted by  $X_1 \times \dots \times X_n$  and the morphism into the product which is induced by the family  $(f_1, \dots, f_n)$  by  $f_1 \times \dots \times f_n$ .

Similarly, we will write  $\text{in}_i: X_i \rightarrow \coprod_{i \in I} X_i$  for the structure morphisms of a coproduct and  $\coprod_{i \in I} g_i: \coprod_{i \in I} X_i \rightarrow Y$  for the morphism from the coproduct which is induced by the family  $(g_i: X_i \rightarrow Y)_{i \in I}$  of morphisms. In the finite case  $X_1 \amalg \dots \amalg X_n$  resp.  $g_1 \amalg \dots \amalg g_n$  will be the alternative notation.

**Notation 0.4.** If a category  $\mathcal{C}$  has a terminal object, we will denote it usually by  $\ast$ . Note that this is compatible with Notation 0.1 for **Cat**. Furthermore, this notation will not cause any ambiguities since it will be clear from context which category  $\ast$  belongs to.

Given an object  $X \in \text{ob } \mathcal{C}$ ,  $\pi_X$  will denote the unique morphism  $X \rightarrow \ast$ .

**Definition 0.5.** If a category  $\mathcal{C}$  has an object which is both initial and terminal, we call  $\mathcal{C}$  *pointed*. Such an object is also called a *zero object* and usually denoted by  $0$ .

## Review of Derivators

Derivators are a way of axiomatizing (homotopy) Kan extensions and hence allow us to describe homotopy theories in purely (2-)categorical terms. Their theory was first developed by Heller (cf. [6]) and Grothendieck (cf. [5]). Later on, the stable case was studied independently by Franke (cf. [2]).

In this subsection, we will briefly review the parts of this theory which we will need in the following sections and try to make some of our slogans into more precise statements. While doing this, we will follow the approach in [3] and [4], where most of the omitted proofs can be found.

We start with the definition of the underlying data of a derivator.

**Definition 0.6.** A *prederivator*  $\mathcal{D}$  is a strict 2-functor from the category  $\mathbf{Cat}$  of small categories to the category  $\mathbf{CAT}$  of all categories which is contravariant on functors (and covariant on natural transformations).

This means that  $\mathcal{D}$  assigns to each small category  $A$  a category  $\mathcal{D}(A)$ , to each functor  $u: A \rightarrow B$  between small categories a functor  $\mathcal{D}(u): \mathcal{D}(B) \rightarrow \mathcal{D}(A)$  and to each natural transformation  $\rho: u \Rightarrow v$  between such functors a natural transformation  $\mathcal{D}(\rho): \mathcal{D}(u) \Rightarrow \mathcal{D}(v)$  such that everything is compatible with compositions and identities.

For the rest of this section  $\mathcal{D}$  will be a prederivator. Furthermore, throughout the thesis we will use a shorter notation for the value of a functor under a prederivator.

**Notation 0.7.** Given a functor  $u: A \rightarrow B$  between small categories, we will write  $u^*$  for  $\mathcal{D}(u): \mathcal{D}(B) \rightarrow \mathcal{D}(A)$  as long as the prederivator in consideration is clear from context.

Here the value  $\mathcal{D}(A)$  (resp.  $\mathcal{D}(B)$ ) can be thought of as the category of “coherent diagrams of shape  $A$  (resp.  $B$ )” and the induced functor  $u^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$  as a “restriction functor” (cf. Example 0.14 and Example 0.15). If one considers  $\mathcal{D}(\ast)$  as “the underlying category of  $\mathcal{D}$ ”, the following remark amounts to considering values of a coherent diagram  $X \in \mathcal{D}(A)$  at a given point  $a \in \text{ob } A$ .

**Remark 0.8.** Given a category  $\mathcal{C}$ , an object  $C \in \text{ob } \mathcal{C}$  corresponds to the functor  $\ast \rightarrow \mathcal{C}$  which sends  $\ast$  to  $C$  and  $\text{id}_\ast$  to  $\text{id}_C$ . Under this identification a natural transformation between two functors  $C, D: \ast \rightarrow \mathcal{C}$  corresponds to a morphism  $C(\ast) \rightarrow D(\ast)$ . Furthermore, this assignment induces an isomorphism of categories between  $\mathcal{C}$  and  $\mathcal{C}^\ast$ .

Hence, given a small category  $A$  and an object  $a$  of  $A$ , one can consider  $a$  as a functor  $a: \ast \rightarrow A$ . This yields a functor  $a^*: \mathcal{D}(A) \rightarrow \mathcal{D}(\ast)$ . We will denote it also by  $(\_)_a$  and hence write  $f_a: X_a \rightarrow Y_a$  for the image of a morphism  $f: X \rightarrow Y$  in  $\mathcal{D}(A)$  under  $a^*$ .

The next step is immitating the description of (homotopy) Kan extensions as adjoints to restriction functors.

**Definition 0.9.** Let  $u: A \rightarrow B$  be a functor between small categories.

- We say that  $\mathcal{D}$  admits left Kan extensions along  $u$  if  $u^*$  has a left adjoint  $u_!$ . In this case we denote the unit of this adjunction by  $\eta_{u_!, u^*}$  and the counit by  $\epsilon_{u_!, u^*}$ .
- We say that  $\mathcal{D}$  admits right Kan extensions along  $u$  if  $u^*$  has a right adjoint  $u_*$ . In this case we denote the unit of this adjunction by  $\eta_{u^*, u_*}$  and the counit by  $\epsilon_{u^*, u_*}$ .

In order to make the definition of derivators more understandable, we will first review the calculus of “mates” which plays a central role in the theory derivators.

**Definition 0.10.** Given functors  $A \xrightarrow{p} B \xrightarrow{u} D$  and  $A \xrightarrow{v} C \xrightarrow{q} D$  between small categories and a natural transformation  $u \circ p \Rightarrow q \circ v$ , applying  $\mathcal{D}$  yields a diagram

$$\begin{array}{ccc} \mathcal{D}(A) & \xleftarrow{p^*} & \mathcal{D}(B) \\ v^* \uparrow & \swarrow & \uparrow u^* \\ \mathcal{D}(C) & \xleftarrow{q^*} & \mathcal{D}(D) \end{array}$$

with a natural transformation  $\alpha: p^* \circ u^* \Rightarrow v^* \circ q^*$ .

Assuming that  $\mathcal{D}$  admits both left and right Kan extensions along all functors involved, the corresponding adjunction units and counits yield two diagrams

$$\begin{array}{ccccccc} \mathcal{D}(C) & \xleftarrow{v_!} & \mathcal{D}(A) & \xleftarrow{p^*} & \mathcal{D}(B) & \xleftarrow{\text{id}_{\mathcal{D}(B)}} & \\ & \swarrow & v^* \uparrow & \swarrow & \uparrow u^* & \swarrow & \\ \text{id}_{\mathcal{D}(C)} & & \mathcal{D}(C) & \xleftarrow{q^*} & \mathcal{D}(D) & \xleftarrow{u_!} & \mathcal{D}(B) \end{array}$$

and

$$\begin{array}{ccc}
\mathcal{D}(B) & \xleftarrow{\text{id}_{\mathcal{D}(B)}} & \mathcal{D}(B) \\
p_* \uparrow & \swarrow & \\
\mathcal{D}(A) & \xleftarrow{p^*} & \mathcal{D}(B) \\
v_* \uparrow & \swarrow & \uparrow u^* \\
\mathcal{D}(C) & \xleftarrow{q^*} & \mathcal{D}(D) \\
& \swarrow & \uparrow q_* \\
& \text{id}_{\mathcal{D}(C)} & \mathcal{D}(C)
\end{array}$$

Hence we obtain natural transformations

$$\alpha_! : v_! \circ p^* \xrightarrow{(v_! \circ p^*) \eta_{u_!, u^*}} v_! \circ p^* \circ u^* \circ u_! \xrightarrow{v_! \alpha u_!} v_! \circ v^* \circ q^* \circ u_! \xrightarrow{\epsilon_{v_!, v^*} (q^* \circ u_!)} q^* \circ u_!$$

and

$$\alpha_* : u^* \circ q_* \xrightarrow{\eta_{p^*, p^*} (u^* \circ q_*)} p_* \circ p^* \circ u^* \circ q_* \xrightarrow{p_* \alpha q_*} p_* \circ v^* \circ q^* \circ q_* \xrightarrow{(p_* \circ v^*) \epsilon_{q^*, q_*}} p_* \circ v^*$$

which we call the *mate transformations associated to  $\alpha$* .

We now summarize some important results about mates.

**Remark 0.11.** In the situation of the previous definition the following statements hold.

- Consider the “degenerate” case where  $\alpha$  is induced by a commutative square

$$\begin{array}{ccc}
A & \xrightarrow{\text{id}_A} & A \\
v \downarrow & \cong & \downarrow v \\
C & \xrightarrow{\text{id}_C} & C,
\end{array}$$

i.e. the natural transformation  $\text{id}_v : v \Rightarrow v$ . Then  $\alpha = \text{id}_{v^*}$  by the 2-functoriality of  $\mathcal{D}$  and  $\alpha_! = \text{id}_{v_!} : v_! \Rightarrow v_!$  by the triangular identity for the left adjoint functor  $v_!$ .

- Similarly, if  $A = C$ ,  $B = D$ ,  $v = \text{id}_A$ ,  $u = \text{id}_B$ ,  $p = q$  and the given natural transformation  $p \Rightarrow p$  is  $\text{id}_p$ , then  $\alpha_* = \text{id}_{p^*}$ .
- Consider a “horizontally adjacent square” as in

$$\begin{array}{ccccc}
A & \xrightarrow{p} & B & \xrightarrow{r} & E \\
v \downarrow & \cong & \downarrow u & \cong & \downarrow t \\
C & \xrightarrow{q} & D & \xrightarrow{s} & F
\end{array}$$

where we call the natural transformation induced by the new square  $\beta: r^* \circ t^* \Rightarrow u^* \circ s^*$ . Then we obtain a *horizontal pasting*

$$\gamma: p^* \circ r^* \circ t^* \xrightarrow{p^* \beta} p^* \circ u^* \circ s^* \xrightarrow{\alpha s^*} v^* \circ q^* \circ s^*.$$

Furthermore,  $\gamma_!$  coincides with the horizontal pasting of  $\alpha_!$  and  $\beta_!$ .

- Similarly, given a “vertically adjacent square” as in

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ v \downarrow & \swarrow & \downarrow u \\ C & \xrightarrow{q} & D \\ w \downarrow & \swarrow & \downarrow t \\ E & \xrightarrow{r} & F \end{array},$$

we obtain a *vertical pasting*  $p^* \circ u^* \circ t^* \Rightarrow v^* \circ w^* \circ r^*$  which is compatible with  $(\_)_{*}$ .

- $\alpha_!$  is an isomorphism if and only if  $\alpha_*$  is an isomorphism.

We need one more concept before we can give a definition of derivators.

**Definition 0.12.** Let  $u: A \rightarrow B$  be a functor between small categories and  $b \in \text{ob } B$ .

Then we obtain two *slice categories*  $(u/b)$  and  $(b/u)$  given by

$$\text{ob}((u/b)) = \{(a, f) \mid a \in \text{ob } A, f: u(a) \rightarrow b\}$$

and

$$\text{Hom}_{(u/b)}((a, f), (a', f')) = \{g \in \text{Hom}_A(a, a') \mid f' = f' \circ u(g)\}$$

respectively

$$\text{ob}((b/u)) = \{(a, f) \mid a \in \text{ob } A, f: b \rightarrow u(a)\}$$

and

$$\text{Hom}_{(b/u)}((a, f), (a', f')) = \{g \in \text{Hom}_A(a, a') \mid f' = u(g) \circ f\}.$$

There is a functor  $p: (u/b) \rightarrow A$  which is given by  $p((a, f)) = a$  on objects and  $p(g) = g$  on morphisms. Similarly, there is a functor  $q: (b/u) \rightarrow A$  given by  $q((a, f)) = a$  on objects and  $q(g) = g$  on morphisms.

Furthermore, there is a natural transformation  $\phi_{u,b}$  from  $u \circ p$  to the constant functor  $b \circ \pi_{(u/b)}$  that is given by  $f: u(p((a, f))) = u(a) \rightarrow b$  for each  $(a, f) \in \text{ob}((u/b))$ . Similarly, there is a natural transformation  $\psi_{u,b}: b \circ \pi_{(b/u)} \Rightarrow u \circ q$  which is given by  $f: b \rightarrow u(a) = u(q((a, f)))$  for  $(a, f) \in \text{ob}((u/b))$ .

Now we have everything at hand to define derivators.

**Definition 0.13.** A prederivator  $\mathcal{D}$  is called a *derivator* if the following conditions are satisfied:

(Der1)  $\mathcal{D}$  takes coproducts to products, i. e. for a family  $(A_i)_{i \in I}$  of small categories the functor

$$\mathcal{D} \left( \coprod_{i \in I} A_i \right) \xrightarrow{\prod_{i \in I} \text{in}_i^*} \prod_{i \in I} \mathcal{D}(A_i)$$

is an equivalence of categories.

(Der2) For  $A \in \text{ob } \mathbf{Cat}$ , a morphism  $f: X \rightarrow Y$  in  $\mathcal{D}(A)$  is an isomorphism iff for each  $a \in \text{ob } A$ ,  $f_a: X_a \rightarrow Y_a$  is an isomorphism in  $\mathcal{D}(\ast)$ .

(Der3) For a functor  $u: A \rightarrow B$  between small categories,  $\mathcal{D}$  has both left and right Kan extensions along  $u$ .

(Der4) For a functor  $u: A \rightarrow B$  between small categories and an object  $b \in B$ , the mate transformation

$$(\Phi_{u,b})! : (\pi_{u/b})! \circ p^* \Rightarrow b^* \circ u!$$

induced by  $\phi_{u,b}$  and the mate transformation

$$(\Psi_{u,b})_* : b^* \circ u_* \Rightarrow (\pi_{(b/u)})_* \circ q^*$$

induced by  $\psi_{u,b}$  are isomorphisms.

Note that the axioms (Der1)-(Der4) do not add any new data to a prederivator  $\mathcal{D}$ , but they merely require  $\mathcal{D}$  to have certain properties. Even (Der3) essentially does not require any choices to be made since adjoint functors are unique up to isomorphism if they exist.

Before we proceed with some useful properties of derivators, we want to consider some examples which are the main motivation for studying derivators. We start with a classical one which is discussed in [7] without using the terminology of derivators.

**Example 0.14.** To each category  $\mathcal{C}$  we can assign its *represented prederivator*  $y_{\mathcal{C}}$  which is given by

- $y_{\mathcal{C}}(A) = \mathcal{C}^A$  on small categories  $A \in \text{ob } \mathbf{Cat}$ ,
- $u^* = \_ \circ u: \mathcal{C}^B \rightarrow \mathcal{C}^A$  on functors  $u: A \rightarrow B$  between small categories
- $(y_{\mathcal{C}}(\alpha))_F = F\alpha: F \circ u \Rightarrow F \circ v$  for  $F \in \mathcal{C}^B$  on natural transformations  $\alpha: u \Rightarrow v$  between such functors.

Note that represented prederivators always satisfy (Der1) and (Der2).

Now assume that  $\mathcal{C}$  is complete and cocomplete. Then, for a functor  $u: A \rightarrow B$ ,  $u^*: \mathcal{C}^B \rightarrow \mathcal{C}^A$  has indeed both a left adjoint and a right adjoint, which are given by classical Kan extensions along  $u$ . This means that (Der3) holds. Furthermore, there are “formulas” for computing these classical Kan extensions pointwise in terms of colimits and limits which correspond to the axiom (Der4).

Another example which is more general and more closely related to homotopy theory is given by model categories. The statement in its full generality is due to Cisinski (cf. [1]).

**Example 0.15.** Let  $\mathcal{M}$  be a model category. Given a small category  $A$ , we denote by  $W_A$  the class of morphisms in  $\mathcal{M}^A$  which are pointwise weak equivalences.

Then one can construct a derivator  $\mathcal{H}o_{\mathcal{M}}$ , called the *homotopy derivator of  $\mathcal{M}$* , which is given by localizations  $\mathcal{M}^A[W_A^{-1}]$  (which can be realized as locally small categories) on objects  $A \in \text{ob } \mathbf{Cat}$  and restriction functors induced by  $_{-} \circ u: \mathcal{M}^B \rightarrow \mathcal{M}^A$  on functors  $u: A \rightarrow B$ .

In particular, one has homotopy derivators associated with chain complexes of modules over a ring, (pointed) topological spaces, (pointed) simplicial sets and spectra.

Let  $\mathcal{D}$  be a derivator for the rest of this section. Then (Der1) and (Der3) yield the following statement.

**Remark 0.16.** As the diagonal functor  $\Delta: \mathcal{D}(\ast) \rightarrow \mathcal{D}(\ast)^I \simeq \mathcal{D}(I)$  has a left adjoint and a right adjoint for any index set  $I$ , coproducts and products exist in  $\mathcal{D}(\ast)$ .

In particular,  $\mathcal{D}(\ast)$  has an initial object  $\emptyset$  and a terminal object  $\ast$  (which is not to be confused with the terminal category  $\ast$ ).

There are certain types of functors  $u: A \rightarrow B$  for which Kan extensions along  $u$  are particularly easy to compute.

**Definition 0.17.** Let  $u: A \rightarrow B$  be a fully faithful functor between small categories.

- $u$  is called a *cosieve* if for every morphism  $u(a) \rightarrow b$  in  $B$  it follows that  $b \in \text{essim } u$ .
- $u$  is called a *sieve* if for every morphism  $b \rightarrow u(a)$  in  $B$  it follows that  $b \in \text{essim } u$ .

Indeed, some calculations yield the following very useful statement.

**Remark 0.18.** If  $u: A \rightarrow B$  is a cosieve, then  $X \in \text{ob}(\mathcal{D}(B))$  is in the essential image of  $u_!$  if and only if  $X_b \cong \emptyset$  for all  $b \in (\text{ob } B) \setminus u(\text{ob } A)$ .

Similarly, if  $u: A \rightarrow B$  is a sieve, then  $X \in \text{ob}(\mathcal{D}(B))$  is in the essential image of  $u_*$  if and only if  $X_b \cong *$  for all  $b \in (\text{ob } B) \setminus u(\text{ob } A)$ .

We end this section with definitions of a few concepts which play an important role in the theory of derivators.

**Definition 0.19.** A square

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ v \downarrow & \lrcorner & \downarrow u \\ C & \xrightarrow{q} & D \end{array}$$

in  $\mathbf{Cat}$  is called *homotopy exact*, if for any derivator  $\mathcal{E}$ , the mate transformations associated with

$$\begin{array}{ccc} \mathcal{E}(A) & \xleftarrow{p^*} & \mathcal{E}(B) \\ v^* \uparrow & \alpha \lrcorner & \uparrow u^* \\ \mathcal{E}(C) & \xleftarrow{q^*} & \mathcal{E}(D) \end{array}$$

are isomorphisms.

Note that in the situation of the previous definition,  $\alpha_!$  is an isomorphism if and only if  $\alpha_*$  is an isomorphism. In particular, (Der4) means that all “slice squares” of the form

$$\begin{array}{ccc} (u/b) & \xrightarrow{p} & A \\ \pi_{(u/b)} \downarrow & \lrcorner & \downarrow u \\ * & \xrightarrow{b} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} (b/u) & \xrightarrow{q} & A \\ \pi_{(b/u)} \downarrow & \lrcorner & \downarrow u \\ * & \xrightarrow{b} & B \end{array}$$

are homotopy exact.

The last definition of this section brings us closer to the title of this thesis.

**Definition 0.20.**  $\mathcal{D}$  is called *pointed* if  $\mathcal{D}(*)$  has a zero object 0.



# 1 Loop Objects

In this section we prove our main result, i. e. show that loop objects in values of pointed derivators are group objects.

Let  $\mathcal{D}$  be a pointed derivator throughout this section.

## Simplicial Objects which Induce Loop Objects

The crucial point of our discussion of the loop objects is the fact that they are induced by certain families of objects which fulfill slightly more general conditions than that for a simplicial object.

**Notation 1.1.** • Let  $\langle n \rangle := \{0, \dots, n\}$  for  $n \in \mathbb{N}$ . We will consider these as objects of the category **Fin** of finite sets or (equivalently) finite discrete categories.

- Let  $\_ \triangleright : \mathbf{Fin} \rightarrow \mathbf{Cat}$  be the cocone functor, i. e. the functor which adds a terminal object  $\infty$  to a given category. Let  $\lrcorner_n := \langle n \rangle^\triangleright$ .

**Definition 1.2.** For  $a \in \mathbf{Fin}$  let  $\omega_a$  be the composition

$$\omega_a : \mathcal{D}(\ast) \xrightarrow{\infty!} \mathcal{D}(a^\triangleright) \xrightarrow{\lim_{a^\triangleright}} \mathcal{D}(\ast).$$

For  $n \in \mathbb{N}$  we will abuse notation and write  $\omega_n$  for  $\omega_{\langle n \rangle}$ . In particular, for the loop functor defined in [4, Definition 8.17] we have

$$\Omega \cong \omega_1 : \mathcal{D}(\ast) \xrightarrow{\infty!} \mathcal{D}(\lrcorner_1) \xrightarrow{\lim_{\lrcorner_1}} \mathcal{D}(\ast).$$

**Lemma 1.3.** *The assignment  $a \mapsto \omega_a$  can be made into a functor*

$$\omega : \mathbf{Fin}^{\text{op}} \rightarrow \text{END}_{\mathbf{CAT}}(\mathcal{D}(\ast)),$$

i. e. for each  $a \in \text{ob } \mathbf{Fin}$ ,  $\omega_a$  is an endofunctor of  $\mathcal{D}(\ast)$  and we can assign each to map  $f : a \rightarrow b$  between finite sets a natural transformation  $\omega_f : \omega_b \Rightarrow \omega_a$  s. t. this assignment is compatible with compositions and identities.

*Proof.* For  $a \in \text{ob } \mathbf{Fin}$ ,  $\omega_a$  is an endofunctor of  $\mathcal{D}(\ast)$  by construction.

For functoriality, we consider  $a, b \in \text{ob } \mathbf{Fin}$  and  $f : a \rightarrow b$ . Then we have two diagrams

$$\begin{array}{ccc} \ast & \longrightarrow & \ast \\ \infty \downarrow & & \downarrow \infty \\ a^\triangleright & \xrightarrow{f^\triangleright} & b^\triangleright \\ \downarrow & & \downarrow \\ \ast & \longrightarrow & \ast \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{D}(\ast) & \longleftarrow & \mathcal{D}(\ast) \\ \infty! \downarrow & \cong & \downarrow \infty! \\ \mathcal{D}(a^\triangleright) & \xleftarrow{(f^\triangleright)^\ast} & \mathcal{D}(b^\triangleright) \\ \lim_{a^\triangleright} \downarrow & \cong & \downarrow \lim_{b^\triangleright} \\ \mathcal{D}(\ast) & \longleftarrow & \mathcal{D}(\ast) \end{array}, \quad (1)$$

where the second one is obtained from the first by applying  $\mathcal{D}$  and then using the appropriate mates.

Now we want to show that the upper natural transformation on the right is an isomorphism and then define the natural transformation  $\omega_f: \omega_b \Rightarrow \omega_a$  as the pasting of the two squares on the right.

Note that we can detect such isomorphisms pointwise. In order to do that, we consider an  $x \in \text{ob}(a)$ , which yields a diagram

$$\begin{array}{ccccc} (\infty/x) & \xrightarrow{\pi} & * & \longrightarrow & * \\ \pi \downarrow & \swarrow & \downarrow \infty & \searrow & \downarrow \infty \\ * & \xrightarrow{x} & a^{\triangleright} & \xrightarrow{f^{\triangleright}} & b^{\triangleright} \end{array}$$

Then we know that the mate transformation  $\pi_! \pi^* \Rightarrow x^* \infty_!$  is an isomorphism since the square on the left is a slice square and hence homotopy exact. Furthermore, we have

$$(\infty/x) \cong \begin{cases} \emptyset & x \neq \infty \\ * & x = \infty \end{cases}.$$

Since  $f^{\triangleright}(x) = \infty$  iff  $x = \infty$ , this yields that the pasting of the two squares is a also slice square, hence homotopy exact, which means that the mate transformation  $\pi_! \pi^* \Rightarrow (f^{\triangleright}(x))^* \infty_!$  is an isomorphism. Hence, in total, we obtain that the mate transformation  $x^* \infty_! \Rightarrow (f^{\triangleright}(x))^* \infty_!$  is an isomorphism.

We can now define  $\omega_f: \omega_b \Rightarrow \omega_a$  to be the pasting of the inverse of  $\infty_! \Rightarrow (f^{\triangleright})^* \infty_!$  with  $\text{lim}_{b^{\triangleright}} \Rightarrow \text{lim}_{a^{\triangleright}} (f^{\triangleright})^*$ . This construction is compatible with composition of maps since mates are compatible with pastings. Furthermore, identities are mapped to identities since all the natural transformations in (1) are identities if  $f$  is an identity map. (See Remark 0.11 for an elaboration of these facts.)  $\square$

**Corollary 1.4.** *For  $X \in \text{ob}(\mathcal{D}(*))$ ,  $(\omega_n X)_{n \in \mathbb{N}}$  can be viewed as a simplicial object since the simplex category  $\mathbf{\Delta}$  is a subcategory of  $\mathbf{Fin}$  (which is not full).*

## Loop Objects as Monoid Objects

Our next step is showing that the simplicial objects associated with loop objects are trivial in the zeroth level and satisfy the Segal condition, which means that loop objects are monoid objects.

**Remark 1.5.** For  $X \in \text{ob}(\mathcal{D}(*))$  we have  $\omega_0 X \cong 0^* \infty_! X$  since 0 is the terminal object of  $\langle 0 \rangle$ , and hence  $\omega_0 \cong 0$  since  $\infty: * \rightarrow \langle 0 \rangle$  is a cosieve.

**Proposition 1.6.** *Let  $n > 1$ . We define  $i_n: \langle n-1 \rangle \rightarrow \langle n \rangle$  to be the inclusion and  $i'_n: \langle 1 \rangle \rightarrow \langle n \rangle$  to be the function with  $i'_n(0) = n-1$  resp.  $i'_n(1) = n$ .*

*Then the natural transformation  $\alpha_n: \omega_n \Rightarrow \omega_{n-1} \times \omega_1$  induced by the functor  $k_n := i_n^{\triangleright} \amalg i_n'^{\triangleright}: \downarrow_{n-1} \amalg \downarrow_1 \rightarrow \downarrow_n$  is an isomorphism.*

*Proof.* Let  $J_n$  be the category which is obtained from  $\downarrow_n$  by adding two objects  $w_0, w_1$  with morphisms  $w_0 \rightarrow k$  for  $0 \leq k \leq n-1$  resp.  $w_1 \rightarrow k$  for  $n-1 \leq k \leq n$  (and resulting compositions), and let  $j_n: \downarrow_n \rightarrow J_n$  denote its inclusion functor.

Let  $\downarrow$  be the full subcategory of  $J_n$  containing  $w_0, w_1$  and  $n-1$  (which is isomorphic to  $\downarrow_1$ ), and let  $l_n$  denote its inclusion functor. Since  $n-1$  is terminal in  $\downarrow$ , we will denote it also by  $\infty$ . Note that  $l_n$  has a right adjoint  $r_n$  given by

$$r_n(x) = \begin{cases} w_0 & x \in \{w_0, 0, \dots, n-2\} \\ w_1 & x \in \{w_1, n\} \\ n-1 & x \in \{n-1, \infty\} \end{cases} .$$

for  $x \in \text{ob } J_n$ , which defines the images of morphisms uniquely. Hence we have  $l_n^* \cong (r_n)_*$ .

Then, using the natural equivalence  $\mathcal{D}(A \amalg B) \simeq \mathcal{D}(A) \times \mathcal{D}(B)$  for  $A, B \in \text{ob } \mathbf{Cat}$  and appropriate mates, we obtain a diagram

$$\begin{array}{ccccc} \mathcal{D}(\ast) & \xrightarrow{(\text{id} \amalg \text{id})^*} & \mathcal{D}(\ast \amalg \ast) & & \\ (\infty_n)! \downarrow & \swarrow & \downarrow (\infty_{n-1} \amalg \infty_1)! & & \\ \mathcal{D}(\downarrow_n) & \xrightarrow{k_n^*} & \mathcal{D}(\downarrow_{n-1} \amalg \downarrow_1) & & \\ (j_n)_* \downarrow & \searrow & \downarrow (\pi_{n-1} \amalg \pi_1)_* & & \\ \mathcal{D}(J_n) & \xrightarrow{(w_0 \amalg w_1)^*} & \mathcal{D}(\ast \amalg \ast) & \xrightarrow{(\pi_{\ast \amalg \ast})_*} & \mathcal{D}(\ast) \\ l_n^* \cong (r_n)_* \downarrow & \searrow & \downarrow (w_0 \amalg w_1)_* \cong & & \downarrow \text{id}_* \\ \mathcal{D}(\downarrow) & \xrightarrow{\text{id}^*} & \mathcal{D}(\downarrow) & \xrightarrow{(\pi_1)_*} & \mathcal{D}(\ast) \end{array} .$$

Under the equivalences mentioned above the upper natural transformation is given by

$$(\infty_{n-1})! \times (\infty_1)! \Rightarrow \left( (i_n^{\triangleright})^* \times (i_n'^{\triangleright})^* \right) (\infty_n)! ,$$

which is the product of the natural transformations which occur in the definition of  $\omega_{i_n}$  resp.  $\omega_{i'_n}$  (cf. (1) in the proof of Lemma 1.3). Hence it is an isomorphism as product of two natural isomorphisms. Furthermore, the right square in the last row commutes up to isomorphism since  $\pi_1(w_0 \amalg w_1) = \pi_{\ast \amalg \ast}$ .

All in all, the diagram above yields a natural transformation from

$$(\pi_1)_*(r_n)_*(j_n)_*(\infty_n)! \cong (\pi_n)_*(\infty_n)! \cong \omega_n$$

to

$$(\pi_{*\Pi*})_*(\pi_{n-1}\amalg\pi_1)_*(\infty_{n-1}\amalg\infty_1)! \cong ((\pi_{n-1})_*(\infty_{n-1})!) \times ((\pi_1)_*(\infty_1)!) \cong \omega_{n-1} \times \omega_1$$

which is the  $\alpha_n$  mentioned in the statement of this proposition. We now want to show that (certain restrictions of) the natural transformations in the remaining two squares are isomorphisms, which will imply that  $\alpha_n$  is an isomorphism.

For the middle square we consider diagrams of the form

$$\begin{array}{ccccc} (x/(\pi_{n-1} \amalg \pi_1)) & \xrightarrow{p_{x, \pi_{n-1} \amalg \pi_1}} & \lrcorner_{n-1} \amalg \lrcorner_1 & \xrightarrow{k_n} & \lrcorner_n \\ \pi_{(x/(\pi_{n-1} \amalg \pi_1))} \downarrow & \cong & \pi_{n-1} \amalg \pi_1 \downarrow & \cong & \downarrow j_n \\ * & \xrightarrow{x} & * \amalg * & \xrightarrow{w_0 \amalg w_1} & J_n \end{array}$$

for  $x \in \text{ob}(* \amalg *) = \{*_0, *_1\}$ .

Then we have

$$(*_0/(\pi_{n-1} \amalg \pi_1)) \cong \lrcorner_{n-1} \quad \text{and} \quad (*_1/(\pi_{n-1} \amalg \pi_1)) \cong \lrcorner_1$$

where under this identification  $p_{*_0, \pi_{n-1} \amalg \pi_1}$  resp.  $p_{*_1, \pi_{n-1} \amalg \pi_1}$  is given by the inclusion  $\iota_0$  resp.  $\iota_1$  of the corresponding category. Since the left square is a slice square, this means that

$$*_0^*(\pi_{n-1} \amalg \pi_1)_* \Rightarrow (\pi_{(*_0/(\pi_{n-1} \amalg \pi_1))})_*(p_{*_0, \pi_{n-1} \amalg \pi_1})^* \cong (\pi_{n-1})_* \iota_0^*$$

and

$$*_1^*(\pi_{n-1} \amalg \pi_1)_* \Rightarrow (\pi_{(*_1/(\pi_{n-1} \amalg \pi_1))})_*(p_{*_1, \pi_{n-1} \amalg \pi_1})^* \cong (\pi_1)_* \iota_1^*$$

are isomorphisms.

On the other hand, we also have

$$(w_0/j_n) \cong \lrcorner_{n-1} \quad \text{and} \quad (w_1/j_n) \cong \lrcorner_1,$$

where under this identification  $p_{w_0, j_n}$  is given by  $i_n^\triangleright = k_n \iota_0$  and  $p_{w_1, j_n}$  is given by  $i_n^{\triangleright} = k_n \iota_1$ . Hence the pasting of the above squares is (up to isomorphisms) also a slice square, so the natural transformations

$$((w_0 \amalg w_1)_*)_*(j_n)_* = w_0^*(j_n)_* \Rightarrow (\pi_{(w_0/j_n)})_*(p_{w_0, j_n})^* \cong (\pi_{n-1})_*(i_n^\triangleright)^*$$

and

$$((w_0 \amalg w_1)_*)_*(j_n)_* = w_1^*(j_n)_* \Rightarrow (\pi_{(w_1/j_n)})_*(p_{w_1, j_n})^* \cong (\pi_1)_*(i_n^{\triangleright})^*$$

are also isomorphisms.

Combining these isomorphisms, we see that

$$\begin{aligned} *_0^*(\pi_{n-1} \amalg \pi_1)_* k_n^* &\cong (\pi_{n-1})_* l_0^* k_n^* \\ &\cong (\pi_{n-1})_*(i_n^{\triangleright})^* \cong w_0^*(j_n)_* \\ &\cong ((w_0 \amalg w_1)_* 0)^*(j_n)_* \cong *_0^*(w_0 \amalg w_1)^*(j_n)_* \end{aligned}$$

and

$$\begin{aligned} *_1^*(\pi_{n-1} \amalg \pi_1)_* k_n^* &\cong (\pi_{n-1})_* l_1^* k_n^* \\ &\cong (\pi_{n-1})_*(i_n^{\triangleright})^* \cong w_1^*(j_n)_* \\ &\cong ((w_0 \amalg w_1)_* 1)^*(j_n)_* \cong *_1^*(w_0 \amalg w_1)^*(j_n)_*. \end{aligned}$$

Since mates are compatible with pastings this means that the natural transformation  $x^*(w_0 \amalg w_1)^*(j_n)_* \Rightarrow x^*(\pi_{n-1} \amalg \pi_1)_* k_n^*$  is an isomorphism for all  $x \in \text{ob}(\ast \amalg \ast)$ , hence it is an isomorphism as isomorphisms can be detected pointwise.

Note that, in general, the natural transformation

$$l_n^* \cong (r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$$

in the first square of the last row is not an isomorphism for all  $X \in \mathcal{D}(J_n)$ . We are going to “fix” this by restricting our attention to  $\text{essim}((j_n)_*(\infty_n)_!)$ .

First, we compute  $(n-1)^*(j_n)_* X'$  for  $X' \in \text{essim}(\infty_!)$ : Consider the slice square

$$\begin{array}{ccc} (n-1/j_n) & \xrightarrow{p} & \lrcorner_n \\ \pi \downarrow & \Rightarrow & \downarrow j_n \\ \ast & \xrightarrow{n-1} & J_n \end{array}$$

Then we know that  $(n-1)^*(j_n)_* \Rightarrow \pi_* p^*$  is an isomorphism.

Now  $(n-1/j_n)$  is isomorphic to the full subcategory  $K_n$  of  $\lrcorner_n$  spanned by  $n-1$  and  $\infty$ , where  $p$  corresponds to the inclusion  $K_n \rightarrow \lrcorner_n$  under this identification. Hence we see that  $\pi_* p^* \cong (n-1)^* p^* \cong (p(n-1))^*$  since  $n-1$  is the initial object of  $K_n$ . Therefore  $(n-1)^*(j_n)_* \cong (n-1)^*$ , where the former  $n-1$  is the object in  $J_n$  and the latter the one in  $\lrcorner_n$ . Since  $\infty: \ast \rightarrow \lrcorner_n$  is a cosieve we know that  $(n-1)^* X' \cong 0$  for  $X' \in \text{essim}(\infty_!)$ , so we obtain  $(n-1)^*(j_n)_* X' \cong 0$ .

This means that for  $X \in \text{essim}((j_n)_*(\infty_n)_!)$  we have  $\infty^* l_n^* X \cong (l_n \infty)^* X \cong (n-1)^* X \cong 0$ . On the other hand, for any  $Y \in \mathcal{D}(\ast \amalg \ast)$ , we have  $\infty^*(w_0 \amalg w_1)_* Y \cong 0$  since  $w_0 \amalg w_1$  is a sieve. Hence  $l_n^* \cong (r_n)_*$  and  $(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$  agree on  $\infty = n-1$  for  $X \in \text{essim}((j_n)_*(\infty_n)_!)$ .

We now consider  $w_i$  for  $i \in \{0, 1\}$ . In the slice square

$$\begin{array}{ccc} (w_i/w_0 \amalg w_1) & \xrightarrow{p} & \ast \amalg \ast \\ \pi \downarrow & \Rightarrow & \downarrow w_0 \amalg w_1 \\ \ast & \xrightarrow{w_i} & \lrcorner \end{array}$$

$(w_i/w_0 \amalg w_1)$  can be identified with  $\ast$  and  $p$  with  $\ast_i: \ast \rightarrow \ast \amalg \ast$ . Hence we see that  $w_i^*(w_0 \amalg w_1)_* \Rightarrow \pi_* w_i^* \cong w_i^*$  is an isomorphism. This yields

$$\begin{aligned} w_i^*(w_0 \amalg w_1)_*(w_0 \amalg w_1)^* &\xrightarrow{\cong} w_i^*(w_0 \amalg w_1)^* = ((w_0 \amalg w_1)w_i)^* \\ &= w_i^* = (l_n w_i)^* = w_i^* l_n^* \cong w_i^*(r_n)_*, \end{aligned}$$

so  $w_i^*(r_n)_* \Rightarrow w_i^*(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$  is an isomorphism by the compatibility of mates with pastings.

All in all,  $(r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$  is an isomorphism pointwise, so it is indeed an isomorphism. This means that the last remaining square is also filled with an isomorphism, so  $\alpha_n: \omega_n \Rightarrow \omega_{n-1} \times w_1$  is an isomorphism in total.  $\square$

**Corollary 1.7.** *Let  $\mu: \langle 1 \rangle \rightarrow \langle 2 \rangle$  be the map with  $\mu(0) = 0$  resp.  $\mu(1) = 2$  and let  $\varepsilon$  be the unique map from  $\langle 1 \rangle$  to  $\langle 0 \rangle$ .*

*Then for any  $X \in \text{ob}(\mathcal{D}(\ast))$ ,  $\omega_1 X \cong \Omega X$  has a monoid object structure given by the multiplication*

$$m_X: \omega_1 X \times \omega_1 X \xrightarrow{(\alpha_2^{-1})_X} \omega_2 X \xrightarrow{(\omega_\mu)_X} \omega_1 X$$

and the unit

$$0: 0 \xrightarrow{\cong} \omega_0 X \xrightarrow{(\omega_\varepsilon)_X} \omega_1 X.$$

*Proof.* The previous proposition and the preceding remark imply that the Segal morphism

$$\omega_n X \rightarrow (\omega_1 X)^n$$

is an isomorphism for any  $n \in \mathbb{N}$ . Then we have  $\omega_0 X \cong 0$  and the simplicial object induced by  $\omega_- X: \mathbf{Fin}^{\text{op}} \rightarrow \mathcal{D}(\ast)$  satisfies the Segal condition, so it is a special simplicial object. Therefore  $\omega_1 X = \Omega X$  has a natural monoid object structure (cf. Proposition A.4).

Now note that  $\langle n \rangle \in \text{ob} \mathbf{Fin}$  and  $[n] \in \text{ob} \mathbf{\Delta}$  are equal as sets for all  $n \in \mathbb{N}$ . Furthermore, we have  $i_2 = \phi^0$ ,  $i'_2 = \phi^1$ ,  $\mu = \delta^1$  and  $\varepsilon = \sigma^0$  as maps between sets. Hence the monoid object structure on  $\omega_1 X$  which is induced by the special simplicial set corresponding to  $\omega_- X$  is indeed given by the morphisms  $m_X$  and  $0$ .  $\square$

## Loop Objects as Group Objects

The last step in this section is the construction of inverses for the multiplication of loop objects, concluding that loop objects are group objects.

**Proposition 1.8.** *Let  $\sigma: \langle 1 \rangle \rightarrow \langle 1 \rangle$  be the only non-trivial automorphism, i. e. the map swapping 0 and 1.*

*Then, for any  $X \in \mathcal{D}(\ast)$ , there is an inversion morphism for the multiplication of  $\Omega X \cong \omega_1 X$  which is given by  $(\omega_\sigma)_X: \omega_1 X \rightarrow \omega_1 X$ .*

*Proof.* We have to show that the composition  $z := m_X \circ (\text{id}_X \times (w_\sigma)_X)$  factors through  $\omega_0 X \cong 0$ , i. e. is the zero morphism. In order to do this we will describe  $z$  as a morphism which factors through  $\omega_2 X$ .

Let  $\phi: \langle 2 \rangle \rightarrow \langle 1 \rangle$  be the map with  $\phi(0) = 0 = \phi(2)$  and  $\phi(1) = 1$ . We claim that the diagram

$$\begin{array}{ccccc}
 & & \omega_2 X & & \\
 & \xrightarrow{(\omega_\phi)_X} & & \xrightarrow{(\omega_\mu)_X} & \\
 & & \downarrow (\alpha_2)_X = \omega_{i_2} \times \omega_{i'_2} & & \\
 \omega_1 X & \xrightarrow{\text{id} \times (\omega_\sigma)_X} & \omega_1 X \times \omega_1 X & \xrightarrow{m_X} & \omega_1 X
 \end{array}$$

commutes.

The right triangle commutes by the definition of  $m_X$ . We verify the commutativity of the left triangle componentwise. Indeed, we have

$$\begin{aligned}
 \text{pr}_1 \circ (\omega_{i_2} \times \omega_{i'_2}) \circ (\omega_\phi)_X &= \omega_{i_2} \circ (\omega_\phi)_X = (\omega_{\phi \circ i_2})_X = (\omega_{\text{id}_{\langle 1 \rangle}})_X \\
 &= \text{id}_{\omega_1 X} = \text{pr}_1 \circ (\text{id}_{\omega_1 X} \times (\omega_\sigma)_X)
 \end{aligned}$$

and

$$\begin{aligned}
 \text{pr}_2 \circ (\omega_{i_2} \times \omega_{i'_2}) \circ (\omega_\phi)_X &= \omega_{i'_2} \circ (\omega_\phi)_X = (\omega_{\phi \circ i'_2})_X = (\omega_\sigma)_X \\
 &= \text{pr}_2 \circ (\text{id}_{\omega_1 X} \times (\omega_\sigma)_X)
 \end{aligned}$$

since  $\phi \circ i_2 = \text{id}_{\langle 1 \rangle}$  and  $\phi \circ i'_2 = \sigma$ .

Hence we obtain that

$$\begin{aligned}
 z &= m_X \circ (\text{id}_X \times (w_\sigma)_X) \\
 &= (\omega_\mu)_X \circ (\alpha_2^{-1})_X \circ (\alpha_2)_X \circ (\omega_\phi)_X \\
 &= (\omega_\mu)_X \circ (\omega_\phi)_X = (\omega_{\phi \circ \mu})_X.
 \end{aligned}$$

Now note that  $\phi \circ \mu$  factors through  $\langle 0 \rangle$  as  $\phi(\mu(0)) = \phi(0) = 0 = \phi(2) = \phi(\mu(0))$ . Hence  $z = (\omega_{\phi \circ \mu})_X$  factors through  $\omega_0 X \cong 0$ .  $\square$

## 2 Double Loop Objects

Our next aim is to show that the group object structure on a double loop object is abelian.

In this section  $\mathcal{D}$  will again be a pointed derivator.

### Loop Functor as a Functor to Group Objects

An important result about the loop functor is the fact that it factors through the category  $\mathcal{D}(\ast)\text{-Grp}$  of group objects in  $\mathcal{D}(\ast)$  also on the level of morphisms.

**Lemma 2.1.** *Let  $f: X \rightarrow Y$  be a morphism in  $\mathcal{D}(\ast)$ .*

*Then the induced morphism  $\Omega f: \Omega X \rightarrow \Omega Y$  is a homomorphism of group objects in  $\mathcal{D}(\ast)$ , where  $\Omega X$  and  $\Omega Y$  are endowed with the group object structure discussed in the previous section.*

*Proof.* First we note that the functors  $\omega_{\_}X, \omega_{\_}Y: \mathbf{Fin}^{\text{op}} \rightarrow \mathcal{D}(\ast)$  induce special simplicial objects as discussed in the previous section. Furthermore, a morphism  $f: X \rightarrow Y$  induces morphisms  $\omega_a f: \omega_a X \rightarrow \omega_a Y$  for  $a \in \text{ob } \mathbf{Fin}$ . This assignment is natural in  $a$  since for a given  $u: a \rightarrow b$ , the diagram

$$\begin{array}{ccc} \omega_a X & \xrightarrow{\omega_a f} & \omega_a Y \\ (\omega_u)_X \uparrow & & \uparrow (\omega_u)_Y \\ \omega_b X & \xrightarrow{\omega_b f} & \omega_b Y \end{array}$$

commutes since  $\omega_u$  is a natural transformation by Lemma 1.3.

Hence  $\omega_{\_}f: \omega_{\_}X \Rightarrow \omega_{\_}Y$  induces a morphism of monoid objects

$$\Omega X = \omega_1 X \xrightarrow{\omega_1 f = \Omega f} \omega_1 Y = \Omega Y$$

as a natural transformation between special simplicial objects (cf. Proposition A.6). Now any morphism of monoid objects between group objects is already a morphism of group objects. (This can be, for example, checked on represented functors and hence can be reduced to the fact that a monoid homomorphism between groups is already a group homomorphism.)  $\square$

### Products under the Loop Functor

The next step in this section is showing that  $\Omega$  preserves products.

**Remark 2.2.** Note that the functor  $\Omega: \mathcal{D}(\ast) \rightarrow \mathcal{D}(\ast)$  has a left adjoint  $\Sigma: \mathcal{D}(\ast) \rightarrow \mathcal{D}(\ast)$  (cf. [4, Proposition 8.18]).

Hence  $\Omega$  preserves limits. In particular, the natural morphism

$$\Omega \left( \prod_{i \in I} X_i \right) \xrightarrow{\prod_{i \in I} \Omega(\text{pr}_i)} \prod_{i \in I} \Omega X_i$$



is an isomorphism for any index set  $I$  and any family  $(X_i)_{i \in I}$  of objects in  $\mathcal{D}(\ast)$ .

This immediately implies that also the group object structure on loop objects are compatible with products.

**Remark 2.3.** For  $X, Y \in \text{ob } \mathcal{D}(\ast)$ , the morphism  $\Omega(X \times Y) \xrightarrow{\Omega(\text{pr}_1) \times \Omega(\text{pr}_2)} \Omega X \times \Omega Y$  is a homomorphism of group objects since it is a product of group object homomorphisms.

This endows  $\Omega(X \times Y)$  with the structure of a product of  $\Omega X$  and  $\Omega Y$  as group objects s. t.

$$m_{\Omega(X \times Y)} = (\Omega(\text{pr}_1) \times \Omega(\text{pr}_2))^{-1} \circ \text{mult}_{\Omega X \times \Omega Y} \circ (\Omega(\text{pr}_1) \times \Omega(\text{pr}_2)),$$

where  $\text{mult}_{\Omega X \times \Omega Y} : (\Omega X \times \Omega Y) \times (\Omega X \times \Omega Y) \rightarrow \Omega X \times \Omega Y$  is the multiplication morphism of the product group object.

Furthermore, the compatibility of  $\Omega$  with products yields a “new” group object structure on double loop objects.

**Corollary 2.4.** For  $X \in \text{ob}(\mathcal{D}(\ast))$ ,  $\Omega^2(X)$  has (in addition to the one given by being the loop object of  $\Omega(X)$ ) a group object structure given by the multiplication

$$m'_X : \Omega^2(X) \times \Omega^2(X) \xrightarrow{\cong} \Omega(\Omega X \times \Omega X) \xrightarrow{\Omega(m_X)} \Omega(\Omega(X)) = \Omega^2 X,$$

the unit

$$0 \rightarrow \Omega^2 X$$

and inverses

$$\Omega^2 X \xrightarrow{\Omega((\omega_\sigma)_X)} \Omega^2 X.$$

*Proof.* The commutativity of the required diagrams follow from the fact that the corresponding diagrams commute before applying  $\Omega$ .  $\square$

## The Eckmann–Hilton Argument

We now have everything at hand to immitate the standard proof of the fact that a group object in **Grp** is an abelian group in order to show that the group object structure on a double loop object is abelian.

**Lemma 2.5.** Let  $X \in \text{ob } \mathcal{D}(\ast)$ . Let  $s_{2,3} := \text{pr}_1 \times \text{pr}_3 \times \text{pr}_2 \times \text{pr}_4 : (\Omega^2 X)^4 \rightarrow (\Omega^2 X)^4$  be the morphism which “swaps the second and the third factor”.

Then the diagram

$$\begin{array}{ccc} \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X & \xrightarrow{s_{2,3}} & \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X \\ \begin{array}{c} \downarrow m'_X \times m'_X \\ \Omega^2 X \times \Omega^2 X \end{array} & & \begin{array}{c} \downarrow m_{\Omega X} \times m_{\Omega X} \\ \Omega^2 X \times \Omega^2 X \end{array} \\ & \searrow m_{\Omega X} & \swarrow m'_X \\ & \Omega^2 X & \end{array}$$

is commutative.

*Proof.* We first note that the diagram

$$\begin{array}{ccc}
& \Omega(\Omega X \times \Omega X) \times \Omega(\Omega X \times \Omega X) & \\
\Omega(m_X) \times \Omega(m_X) \swarrow & & \searrow m_{\Omega \times \Omega} \\
\Omega(\Omega X) \times \Omega(\Omega X) & & \Omega(\Omega X \times \Omega X) \\
m_{\Omega X} \searrow & & \swarrow \Omega(m_X) \\
& \Omega(\Omega X) & 
\end{array} \tag{2}$$

commutes since  $\Omega(m_X): \Omega(\Omega X \times \Omega X) \rightarrow \Omega(\Omega(X))$  is a homomorphism of group objects by Lemma 2.1.

Now  $(m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3}$  is the multiplication morphism of  $\Omega^2 X \times \Omega^2 X$ , which also coincides with  $(\Omega(\text{pr}_1) \times \Omega(\text{pr}_2)) \circ m_{\Omega X \times \Omega X} \circ (\Omega(\text{pr}_1) \times \Omega(\text{pr}_2))^{-1}$  by Remark 2.3.

Hence, identifying  $\Omega(\Omega X \times \Omega X)$  with  $\Omega^2 X \times \Omega^2 X$ , the diagram (2) becomes a commutative diagram

$$\begin{array}{ccc}
& \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X & \\
& \cong \uparrow & \xrightarrow{(m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3}} \\
& \Omega(\Omega X \times \Omega X) \times \Omega(\Omega X \times \Omega X) & \Omega^2 X \times \Omega^2 X \\
& \Omega(m_X) \times \Omega(m_X) \swarrow & \searrow m_{\Omega \times \Omega} \\
\Omega(\Omega X) \times \Omega(\Omega X) & & \Omega(\Omega X \times \Omega X) \\
m_{\Omega X} \searrow & & \swarrow \Omega(m_X) \\
& \Omega(\Omega X) & 
\end{array}$$

which contains the required diagram.  $\square$

**Corollary 2.6.** *The “group laws”  $m_{\Omega X}$  and  $m'_X$  on  $\Omega^2 X$  coincide and are abelian.*

*In particular,  $\Omega^2: \mathcal{D}(\ast) \rightarrow \mathcal{D}(\ast)$  factors through the category  $\mathcal{D}(\ast)\text{-Ab}$  of abelian group objects in  $\mathcal{D}(\ast)$  since each homomorphism of group objects between abelian group objects is a homomorphism of abelian group objects and vice versa.*

*Proof.* Consider the morphism

$$f := \text{pr}_1 \times 0 \times 0 \times \text{pr}_2: \Omega^2 X \times \Omega^2 X \rightarrow \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X.$$

Then we have  $\text{pr}_1 \circ (m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3} \circ f = m_{\Omega X} \circ (\text{pr}_1 \times 0) = \text{pr}_1$  and  $\text{pr}_2 \circ (m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3} \circ f = m_{\Omega X} \circ (\text{pr}_2 \times 0) = \text{pr}_2$  since  $0 \rightarrow \Omega^2 X$  is the

unit morphism for  $m_{\Omega_X}$ . Hence we have  $(m_{\Omega_X} \times m_{\Omega_X}) \circ s_{2,3} \circ f = \text{id}_{\Omega^2 X \times \Omega^2 X}$  as these morphisms agree after composing with each of the projections.

Furthermore, we also have  $\text{pr}_1 \circ (m'_X \times m'_X) \circ f = m'_X \circ (\text{pr}_1 \times 0) = \text{pr}_1$  and  $\text{pr}_2 \circ (m'_X \times m'_X) \circ f = m'_X \circ (\text{pr}_2 \times 0) = \text{pr}_2$  since  $0 \rightarrow \Omega^2 X$  is also the unit morphism for  $m'_X$ . Hence  $(m'_X \times m'_X) \circ f = \text{id}_{\Omega^2 X \times \Omega^2 X}$  as these agree after composing with each of the projections.

In total, using the Eckmann–Hilton identity from the previous lemma, we obtain

$$\begin{aligned} m_{\Omega X} &= m_{\Omega X} \circ \text{id}_{\Omega^2 X \times \Omega^2 X} \\ &= m_{\Omega X} \circ (m'_X \times m'_X) \circ f \\ &= m'_X \circ (m_{\Omega_X} \times m_{\Omega_X}) \circ s_{2,3} \circ f \\ &= m'_X \circ \text{id}_{\Omega^2 X \times \Omega^2 X} = m'_X. \end{aligned}$$

For the commutativity of  $m_{\Omega X} = m'_X$  we consider the morphism

$$g := 0 \times \text{pr}_1 \times \text{pr}_2 \times 0: \Omega^2 X \times \Omega^2 X \rightarrow \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X.$$

Then we have  $\text{pr}_1 \circ (m_{\Omega_X} \times m_{\Omega_X}) \circ s_{2,3} \circ g = m_{\Omega_X} \circ (0 \times \text{pr}_2) = \text{pr}_2$  and  $\text{pr}_2 \circ (m_{\Omega_X} \times m_{\Omega_X}) \circ s_{2,3} \circ g = m_{\Omega_X} \circ (\text{pr}_1 \times 0) = \text{pr}_1$ , therefore  $(m_{\Omega_X} \times m_{\Omega_X}) \circ s_{2,3} \circ g = \text{pr}_2 \times \text{pr}_1$ , i. e. the “swapping morphism”. On the other hand, we also have  $\text{pr}_1 \circ (m'_X \times m'_X) \circ g = m'_X \circ (0 \times \text{pr}_1) = \text{pr}_1$  and  $\text{pr}_2 \circ (m'_X \times m'_X) \circ g = m'_X \circ (\text{pr}_2 \times 0) = \text{pr}_2$ , so  $(m'_X \times m'_X) \circ g = \text{id}_{\Omega^2 X \times \Omega^2 X}$ .

Hence, the Eckmann–Hilton identity yields

$$\begin{aligned} m_{\Omega X} &= m_{\Omega X} \circ \text{id}_{\Omega^2 X \times \Omega^2 X} \\ &= m_{\Omega X} \circ (m'_X \times m'_X) \circ g \\ &= m'_X \circ (m_{\Omega_X} \times m_{\Omega_X}) \circ s_{2,3} \circ g \\ &= m'_X \circ (\text{pr}_2 \times \text{pr}_1) \\ &= m_{\Omega X} \circ (\text{pr}_2 \times \text{pr}_1), \end{aligned}$$

which means that  $m_{\Omega X} = m'_X$  is indeed a commutative multiplication.  $\square$

### 3 Applications

Besides the intrinsic motivation for studying it, the loop functor can be used to show that values of a certain type of derivators are additive.

**Definition 3.1.** A pointed derivator  $\mathcal{D}$  is called *stable* if the loop functor  $\Omega: \mathcal{D}(\ast) \rightarrow \mathcal{D}(\ast)$  is an equivalence of categories.

Typical examples of stable derivators include the homotopy derivator of chain complexes of modules over a ring and the homotopy derivator of spectra (cf. [4]).

Now the statements we have proven about the loop functor yields the following additivity result.

**Corollary 3.2.** *Let  $\mathcal{D}$  be a stable derivator. Then  $\mathcal{D}(\ast)$  is an additive category.*

*Proof.* First of all,  $\mathcal{D}(\ast)$  has all products since  $\mathcal{D}$  is a derivator. Furthermore, we know that  $\Omega$  (and hence  $\Omega^2$ ) is an equivalence of categories since  $\mathcal{D}$  is stable.

Now note that  $\Omega^2: \mathcal{D}(\ast) \rightarrow \mathcal{D}(\ast)$  factors through the category  $\mathcal{D}(\ast)\text{-Ab}$  of abelian group objects in  $\mathcal{D}(\ast)$  by Corollary 2.6 since morphisms between abelian group objects are simply homomorphisms of underlying group objects.

Hence  $\mathcal{D}(\ast) \simeq \mathcal{D}(\ast)\text{-Ab}$ . Now the category of abelian group objects in a category with finite products is an additive category (cf. Corollary B.10).  $\square$

Furthermore, note that one can construct the “shifted derivator”  $\mathcal{D}^A$  for a given small category  $A$ , which is given by  $\mathcal{D}^A(B) = \mathcal{D}(A \times B)$  on small categories,  $(u^\ast)^{\mathcal{D}^A} = (\text{id}_A \times u)^\mathcal{D}$  on functors and  $(\gamma^\ast)^{\mathcal{D}^A} = (\text{id}_{\text{id}_A} \times \gamma)^\mathcal{D}$  on natural transformations (cf. [4, Proposition 7.32]). Then  $\mathcal{D}^A$  is pointed resp. stable if  $\mathcal{D}$  is so, hence we can obtain statements about  $\mathcal{D}(A) \simeq \mathcal{D}^A(\ast)$  by considering  $\mathcal{D}^A$  as a derivator.

**Remark 3.3.** Let  $A$  be a small category.

Then the shifted loop functor

$$\Omega^A := (\text{id}_A \times \pi_\perp)_\ast \circ (\text{id}_A \times \infty)_! : \mathcal{D}(A) \rightarrow \mathcal{D}(A)$$

factors through  $\mathcal{D}(A)\text{-Grp}$  and the double shifted loop functor  $(\Omega^A)^2$  factors even through  $\mathcal{D}(A)\text{-Ab}$ .

Moreover,  $\mathcal{D}(A)$  is an additive category if  $\mathcal{D}$  (and hence  $\mathcal{D}^A$ ) is stable.

## A The Segal Condition

In this appendix we will justify Corollary 1.7 by showing that a certain type of simplicial objects give rise to monoid objects.

We start with a review of simplicial objects.

**Notation A.1.** Let  $\Delta$  be the simplex category, i. e. the category of non-empty finite ordinal numbers. For  $n \in \mathbb{N}$  set  $[n] = \{0, \dots, n\}$ .

For  $n \in \mathbb{N}$  and  $i \in \{0, \dots, n\}$  we fix notation for the following morphisms in  $\Delta$ :

- $\delta^{n,i}: [n-1] \rightarrow [n]$ ,  $n > 0$ , is the unique monomorphism which “skips  $i$ ”,
- $\sigma^{n,i}: [n+1] \rightarrow [n]$  is the unique epimorphism which “collapses  $i+1$  to  $i$ ”,
- $\phi^{n,i}: [1] \rightarrow [n]$ ,  $i < n$ , is the inclusion of  $\{i, i+1\}$ .

In most cases, we will omit the index  $n$  if it is clear from the context.

Given a category  $\mathcal{C}$  and a simplicial object  $X: \Delta^{\text{op}} \rightarrow \mathcal{C}$ , we will denote  $X([n])$  by  $X_n$ . Then the above maps induce:

- $d_i^X := X(\delta^i): X_n \rightarrow X_{n-1}$ , the  $i$ -th face map,
- $s_i^X := X(\sigma^i): X_n \rightarrow X_{n+1}$ , the  $i$ -th degeneracy map,
- $f_i^X := X(\phi^i): X_n \rightarrow X_1$ .

The simplicial object in consideration will mostly be clear from the context and we will omit the upper index  $X$  in these cases.

In a certain sense, face maps and degeneracy maps determine a simplicial object uniquely.

**Remark A.2.** All morphisms in  $\Delta$  can be written as a composition of suitable  $\delta^i$ 's and  $\sigma^i$ 's. These maps satisfy the *simplicial relations*:

- $\delta^j \circ \delta^i = \delta^i \circ \delta^{j-1}$  for  $i < j$ ,
- $\sigma^j \circ \delta^i = \delta^i \circ \sigma^{j-1}$  for  $i < j$ ,
- $\sigma^j \circ \delta^i = \text{id}$  for  $i = j$  and  $i = j + 1$ ,
- $\sigma^j \circ \delta^i = \delta^{i-1} \circ \sigma^j$  for  $i > j$ ,
- $\sigma^j \circ \sigma^i = \sigma^{i-1} \circ \sigma^j$  for  $i > j$ .

Furthermore, all relations between the  $\delta^i$ 's and the  $\sigma^i$ 's are implied by these relations in the following sense:

For a category  $\mathcal{C}$ , a collection  $(X_n)_{n \in \mathbb{N}}$  of objects in  $\mathcal{C}$  with morphisms  $d_i: X_{n-1} \rightarrow X_n$  for  $n > 0$ ,  $0 \leq i \leq n$  and  $s_i: X_n \rightarrow X_{n+1}$  for  $0 \leq i \leq n$  yields a simplicial object  $X$  s. t.  $d_i = X(\delta^i)$  and  $s_i = X(\sigma^i)$  iff the *simplicial identities* (which are induced by the simplicial relations) hold:

- $d_i \circ d_j = d_{j-1} \circ d_i$  for  $i < j$ ,
- $d_i \circ s_j = s_{j-1} \circ d_i$  for  $i < j$ ,
- $d_i \circ s_j = \text{id}$  for  $i = j$  and  $i = j + 1$ ,
- $d_i \circ s_j = s_j \circ d_{i-1}$  for  $i > j$ ,
- $s_i \circ s_j = s_j \circ s_{i-1}$  for  $i > j$ .

A relevant fact in the theory of simplicial sets is that a simplicial set  $X: \mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Set}$  is isomorphic to the nerve of a (small) category if and only if the *Segal condition* is satisfied, i. e. for any  $n \in \mathbb{N}$ , the natural map

$$X_n \xrightarrow{\prod_{i=0}^{n-1} f_i} X_1^{\times_{X_0} n}$$

is an isomorphism. Since (small) categories with only one object can be identified with monoids, this means that simplicial sets  $X$  which have exactly one 0-simplex (i. e.  $X_0 \cong \{*\}$ ) and fulfill the Segal condition can also be identified with monoids.

In the following we want to prove a similar statement for simplicial objects in a category. We know that, in general, fiber products don't exist in values of a derivator, but products do. Therefore we will restrict our attention to simplicial objects  $X$  with  $X_0 \cong *$  for a terminal object  $*$ , so that we have  $X_1^n \cong X_1^{\times_{X_0} n}$ , which makes calculations easier.

In the rest of this appendix  $\mathcal{C}$  will be a category which has all finite products (hence an terminal object  $*$ ) and  $X: \mathbf{\Delta}^{\text{op}} \rightarrow \mathcal{C}$  a simplicial object in  $\mathcal{C}$ .

**Definition A.3.**  $X$  is called *special* if  $X_0 \cong *$  and  $X$  satisfies the Segal condition, i. e.  $X_n \xrightarrow{\prod_{i=0}^{n-1} f_i} X_1^n$  is an isomorphism for all  $n \in \mathbb{N}$ .

We will denote the category of special simplicial objects in  $\mathcal{C}$  with natural transformations between those as morphisms by  $(\mathbf{s}\mathcal{C})^{\text{sp}}$ .

First we show that special simplicial objects give rise to monoid objects.

**Proposition A.4.** *Let  $X$  be a special simplicial object in  $\mathcal{C}$ .*

*Then  $X_1$  has a monoid object structure given by the multiplication morphism*

$$m_X: X_1 \times X_1 \xrightarrow{(f_0 \times f_1)^{-1}} X_2 \xrightarrow{d_1} X_1$$

and the unit morphism

$$e_X: * \xrightarrow{\cong} X_0 \xrightarrow{s_0} X_1,$$

where we will omit the index  $X$  if it is clear from the context.

*Proof.* For associativity we consider the diagram

$$\begin{array}{ccccc}
& & X_1 \times X_1 \times X_1 & & \\
& \swarrow^{(f_0 \circ \text{pr}_1) \times (f_1 \circ \text{pr}_1) \times \text{pr}_2} & \uparrow & \nwarrow^{\text{pr}_1 \times (f_0 \circ \text{pr}_2) \times (f_1 \circ \text{pr}_2)} & \\
X_2 \times X_1 & & X_3 & & X_1 \times X_2 \\
\downarrow^{(d_1 \circ \text{pr}_1) \times \text{pr}_2} & \swarrow^{f_0 \times f_1 \times f_2} & \uparrow & \nwarrow^{f_0 \times d_0} & \downarrow^{\text{pr}_1 \times (d_1 \circ \text{pr}_2)} \\
X_1 \times X_1 & & X_3 & & X_1 \times X_1 \\
\uparrow^{f_0 \times f_1} & \swarrow^{d_3 \times f_2} & \downarrow^{d_1} & \searrow^{d_2} & \uparrow^{f_0 \times f_1} \\
X_2 & & X_2 & & X_2 \\
& \searrow^{d_1} & & \swarrow^{d_1} & \\
& & X_1 & & 
\end{array}$$

The lower parallelogram commutes as  $d_1 \circ d_2 = d_1 \circ d_1$  is one of the simplicial identities. Note that the upper left and upper right sides of the diagram are symmetric, so we will only show that the upper left part is commutative since the commutativity of the other part can be shown similarly.

For the upper left triangle we have

$$\begin{aligned}
\text{pr}_1 \circ ((f_0 \circ \text{pr}_1) \times (f_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2) &= (f_0 \circ \text{pr}_1) \circ (d_3 \times f_2) \\
&= f_0 \circ d_3 = f_0 \\
&= \text{pr}_1 \circ (f_0 \times f_1 \times f_2)
\end{aligned}$$

since  $\phi^0 = \delta^3 \circ \phi^0: [1] \rightarrow [2] \rightarrow [3]$ . Similarly, we also have

$$\begin{aligned}
\text{pr}_2 \circ ((f_0 \circ \text{pr}_1) \times (f_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2) &= f_1 \circ d_3 \\
&= f_1 = \text{pr}_2 \circ (f_0 \times f_1 \times f_2)
\end{aligned}$$

since  $\phi^1 = \delta^3 \circ \phi^1: [1] \rightarrow [2] \rightarrow [3]$ . For the third factor we have

$$\begin{aligned}
\text{pr}_3 \circ ((f_0 \circ \text{pr}_1) \times (f_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2) &= \text{pr}_2 \circ (d_3 \times f_2) \\
&= f_2 = \text{pr}_3 \circ (f_0 \times f_1 \times f_2).
\end{aligned}$$

Hence  $((f_0 \circ \text{pr}_1) \times (f_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2) = f_0 \times f_1 \times f_2$  since these morphisms coincide after composing with each of the projections.

For the middle left triangle we have

$$\begin{aligned}
\text{pr}_1 \circ ((d_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2) &= (d_1 \circ \text{pr}_1) \circ (d_3 \times f_2) \\
&= d_1 \circ d_3 = f_0 \circ d_1 = \text{pr}_1 \circ (f_0 \times f_1) \circ d_1
\end{aligned}$$

since  $\delta^1 \circ \phi^0 = \delta^3 \circ \delta^1: [1] \rightarrow [2] \rightarrow [3]$ . We also have

$$\begin{aligned} \text{pr}_2 \circ ((d_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2) &= \text{pr}_2 \circ (d_3 \times f_2) \\ &= f_2 = f_1 \circ d_1 = \text{pr}_2 \circ (f_0 \times f_1) \circ d_1 \end{aligned}$$

since  $\phi^2 = \delta^1 \circ \phi^1: [1] \rightarrow [2] \rightarrow [3]$ . Hence the morphisms  $((d_1 \circ \text{pr}_1) \times \text{pr}_2) \circ (d_3 \times f_2)$  and  $(f_0 \times f_1) \circ d_1$  coincide as morphisms into the product  $X_1 \times X_1$ .

Inverting the isomorphisms  $f_0 \times f_1 \times f_2$ ,  $(f_0 \circ \text{pr}_1) \times (f_1 \circ \text{pr}_1) \times \text{pr}_2$ ,  $\text{pr}_1 \times (f_0 \circ \text{pr}_2) \times (f_1 \circ \text{pr}_2)$  and  $f_0 \times f_1$ , we obtain a commutative diagram

$$\begin{array}{ccccc} & & X_1 \times X_1 \times X_1 & & \\ & \swarrow \cong & \downarrow \cong & \searrow \cong & \\ X_2 \times X_1 & & & & X_1 \times X_2 \\ \downarrow (d_1 \circ \text{pr}_1) \times \text{pr}_2 & \swarrow d_3 \times f_2 & \downarrow & \searrow f_0 \times d_0 & \downarrow \text{pr}_1 \times (d_1 \circ \text{pr}_2) \\ X_1 \times X_1 & & X_3 & & X_1 \times X_1 \\ \downarrow \cong & \swarrow d_1 & & \searrow d_2 & \downarrow \cong \\ X_2 & & & & X_2 \\ & \swarrow d_1 & & \searrow d_1 & \\ & & X_1 & & \end{array} ,$$

which contains the required associativity diagram.

In order to show that  $e$  is a right unit for  $m$ , we consider the diagram

$$\begin{array}{ccccc} & & X_2 & & \\ & \swarrow s_1 & \downarrow f_0 \times f_1 & \searrow d_1 & \\ X_1 & \xrightarrow{\text{id} \times (e \circ \pi)} & X_1 \times X_1 & \xrightarrow{m} & X_1 \end{array} .$$

Then the triangle on the right commutes by the definition of  $m$ .

On the left side we have

$$\text{pr}_1 \circ (f_0 \times f_1) \circ s_1 = f_0 \circ s_1 = \text{id}_{X_1}$$

since  $\text{id}_{[1]} = \sigma^1 \circ \phi^0: [1] \rightarrow [2] \rightarrow [1]$ . For the second factor we have

$$\text{pr}_2 \circ (f_0 \times f_1) \circ s_1 = f_1 \circ s_1 = s_0 \circ d_1$$

since  $\delta^1 \circ \sigma^0 = \sigma^1 \circ \phi^1: [1] \rightarrow [1]$ , and  $s_0 \circ d_1 = e \circ \pi$  since  $\pi$  is the unique map into the terminal object  $\ast$  and  $e: \ast \xrightarrow{\cong} X_0 \xrightarrow{s_0} X_1$  by definition. Hence the left triangle is also commutative since  $(f_0 \times f_1) \circ s_1$  and  $\text{id}_{X_1} \times (e \circ \pi)$  agree on both factors.

The commutativity of the above diagram yields  $m \circ (\text{id}_{X_1} \times (e \circ \pi)) = d_1 \circ s_1 = \text{id}_{X_1}$ , where the latter equality is a simplicial identity. Hence  $e$  is indeed a right unit for  $m$ .



Now one can analogously show that the diagram

$$\begin{array}{ccccc}
 & & X_2 & & \\
 & \nearrow s_2 & \downarrow f_0 \times f_1 & \searrow d_1 & \\
 X_1 & \xrightarrow{(e \circ \pi) \times \text{id}} & X_1 \times X_1 & \xrightarrow{m} & X_1
 \end{array}$$

is also commutative. Hence  $e$  is also a left unit for  $m$ .  $\square$

Next, we want to see that this assignment is functorial.

**Lemma A.5.** *Let  $X, Y$  be special simplicial objects in  $\mathcal{C}$ , and  $\gamma: X \rightarrow Y$  a morphism of simplicial objects, i. e. a natural transformation between the functors  $X, Y: \Delta^{\text{op}} \rightarrow \mathcal{C}$ .*

*Then we have*

$$\prod_{i=0}^{n-1} (\gamma_1 \circ \text{pr}_i) = \left( \prod_{i=0}^{n-1} f_i^Y \right) \circ \gamma_n \circ \left( \prod_{i=0}^{n-1} f_i^X \right)^{-1} : X_1^n \rightarrow X_n \rightarrow Y_n \rightarrow Y_1^n$$

for all  $n \in \mathbb{N}$ .

*Proof.* For  $n = 0$  the statement follows from the fact that there is a unique morphism between the terminal objects  $X_0$  and  $Y_0$ .

For  $n > 0$  we check the equality componentwise. Indeed, for  $i_0 \in \{0, \dots, n-1\}$  we have

$$\begin{aligned}
 \text{pr}_{i_0} \circ \left( \prod_{i=0}^{n-1} f_i^Y \right) \circ \gamma_n \circ \left( \prod_{i=0}^{n-1} f_i^X \right)^{-1} &= f_{i_0}^Y \circ \gamma_n \circ \left( \prod_{i=0}^{n-1} f_i^X \right)^{-1} \\
 &= \gamma_1 \circ f_{i_0}^X \circ \left( \prod_{i=0}^{n-1} f_i^X \right)^{-1} \\
 &= \gamma_1 \circ \text{pr}_{i_0} \\
 &= \text{pr}_{i_0} \circ \left( \prod_{i=0}^{n-1} (\gamma_1 \circ \text{pr}_i) \right),
 \end{aligned}$$

where the second equality follows from the naturality of  $\gamma$ .  $\square$

**Proposition A.6.** *Let  $\gamma: X \rightarrow Y$  be a morphism between special simplicial objects.*

*Then  $\gamma_1: X_1 \rightarrow Y_1$  is a morphism of monoid objects.*

*Proof.* By the naturality of  $\gamma$ , the diagram

$$\begin{array}{ccc}
 * & \xrightarrow{\text{id}} & * \\
 \cong \downarrow & & \downarrow \cong \\
 X_0 & \xrightarrow{\gamma_0} & Y_0 \\
 s_0^X \downarrow & & \downarrow s_0^Y \\
 X_1 & \xrightarrow{\gamma_1} & Y_0
 \end{array}$$

commutes, so  $\gamma_1$  is compatible with the unit.

For the compatibility with the multiplication we consider the diagram

$$\begin{array}{ccccc} X_1 \times X_1 & \xrightarrow{(f_0^X \times f_1^X)^{-1}} & X_2 & \xrightarrow{\gamma_2} & Y_2 & \xrightarrow{f_0^Y \times f_1^Y} & Y_2 \times Y_1 \\ & \searrow & \downarrow d_1^X & & \downarrow d_1^Y & & \swarrow \\ & & X_1 & \xrightarrow{\gamma_1} & Y_1 & & \\ & \swarrow m_X & & & & & \searrow m_Y \end{array},$$

which is commutative by the naturality of  $\gamma$  and the definition of  $m_X$  resp.  $m_Y$ . Now

$$(f_0^Y \times f_1^Y) \circ \gamma_2 \circ (f_0^X \times f_1^X)^{-1} = (\gamma_1 \circ \text{pr}_1) \times (\gamma_1 \circ \text{pr}_2),$$

by the previous lemma, which implies that the above diagram witnesses the compatibility of  $\gamma_1$  with the multiplication.  $\square$

Proposition A.4 and Proposition A.6 can be summarized as follows:

**Corollary A.7.** *The functor  $(\_)_1: (\mathbf{s}\mathcal{C})^{\text{sp}} \rightarrow \mathcal{C}$  factors through the category  $\mathcal{C}\text{-Mon}$  of monoid objects in  $\mathcal{C}$ .*

In fact,  $(\_)_1$  defines an equivalence of categories from  $(\mathbf{s}\mathcal{C})^{\text{sp}}$  to  $\mathcal{C}\text{-Mon}$ , where a quasi-inverse is given as follows:

For  $M \in \text{ob}(\mathcal{C}\text{-Mon})$  with “multiplication”  $m: M \times M \rightarrow M$  and “unit”  $e: \ast \rightarrow M$ , we define a (special) simplicial object  $X^M$  with  $X_n^M = M^n$  for  $n \in \mathbb{N}$ , where the structure morphisms are given by

$$d_i^{X^M} = \begin{cases} \prod_{j=2}^n \text{pr}_j & i = 0 \\ \prod_{j=0}^{n-1} \text{pr}_j & i = n \\ \left( \prod_{j=1}^{i-2} \text{pr}_j \right) \times m \times \left( \prod_{j=i+1}^n \text{pr}_j \right) & \text{otherwise} \end{cases}$$

for  $n \in \mathbb{N}_{>0}$  and  $0 \leq i \leq n$  resp.

$$s_i^{X^M} = \left( \prod_{j=1}^i \text{pr}_j \right) \times (e \circ \pi) \times \left( \prod_{j=i+1}^n \text{pr}_j \right)$$

for  $n \in \mathbb{N}$ .

Given a morphism  $f: M \rightarrow N$  of monoid objects in  $\mathcal{C}$ , we let  $\gamma^f: X^M \Rightarrow X^N$  be given by

$$\gamma_n^f: X_n^M = M^n \xrightarrow{\prod_{i=0}^n (f \circ \text{pr}_i)} N^n = X_n^N$$

for all  $n \in \mathbb{N}$ .

Since we don't use this statement we omit the tedious proof of the fact that the given assignment is a well-defined functor which is indeed a quasi-inverse for  $(\_)_1: (\mathbf{s}\mathcal{C})^{\text{sp}} \rightarrow \mathcal{C}\text{-Mon}$ .

## B Additive Categories

In this appendix we discuss certain descriptions of additive categories which lead to the additivity of stable derivators (see Corollary 3.2).

We begin with basic definitions and notations.

**Definition B.1.** A *preadditive category* is a category  $\mathcal{A}$  s. t.

- $\mathcal{A}$  is pointed,
- binary (and hence all finite) products and coproducts exist in  $\mathcal{A}$ ,
- for any  $X, Y \in \text{ob } \mathcal{A}$ , the morphism

$$(\text{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \text{id}_Y): X \amalg Y \rightarrow X \times Y$$

is an isomorphism, where  $0_{X,Y}: X \rightarrow 0 \rightarrow Y$  resp.  $0_{Y,X}: Y \rightarrow 0 \rightarrow X$  is the unique morphism which factors through a zero object.

**Notation B.2.** • Biproducts in the above sense will be denoted by  $\_ \oplus \_$ .

- If  $X, Y, X'$  resp.  $Y'$  are objects of a preadditive category and  $f_{X,X'}: X \rightarrow X'$ ,  $f_{Y,X'}: Y \rightarrow X'$ ,  $f_{X,Y'}: X \rightarrow Y'$  resp.  $f_{Y,Y'}: Y \rightarrow Y'$  are some morphisms, then we denote the morphism

$$(f_{X,X'} \times f_{X,Y'}) \amalg (f_{Y,X'} \times f_{Y,Y'}): X \oplus Y \rightarrow X' \oplus Y'$$

by

$$\begin{pmatrix} f_{X,X'} & f_{Y,X'} \\ f_{X,Y'} & f_{Y,Y'} \end{pmatrix}.$$

Note that, using the universal properties of products and coproducts, any morphism  $f: X \oplus Y \rightarrow X' \oplus Y'$  can be written as

$$f = \begin{pmatrix} \text{pr}_{X'} \circ f \circ \text{in}_X & \text{pr}_{X'} \circ f \circ \text{in}_Y \\ \text{pr}_{Y'} \circ f \circ \text{in}_X & \text{pr}_{Y'} \circ f \circ \text{in}_Y \end{pmatrix}.$$

Matrices of different sizes are constructed similarly.

- We will also use common abuses of notation such as denoting an identity morphism by 1 or a morphism that factors through a zero object by 0.

Next, we want to give an alternative description of preadditive categories.

**Remark B.3.** A preadditive category is enriched over the category **AbMon** of abelian monoids. Indeed, for any  $X, Y \in \text{ob } \mathcal{A}$ , setting

$$f + g: X \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \oplus Y \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} Y$$

for  $f, g \in \text{Hom}_{\mathcal{A}}(X, Y)$  yields an abelian monoid structure on  $\text{Hom}_{\mathcal{A}}(X, Y)$  with neutral element  $0_{X,Y}$  and for any  $X, Y, Z \in \text{ob } \mathcal{A}$ , the composition map

$$- \circ -: \text{Hom}_{\mathcal{A}}(Y, Z) \times \text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear w. r. t. this “addition operation”.

Furthermore, a straightforward computation shows that composing morphisms corresponds to multiplying their matrix representations.

**Proposition B.4.** *Let  $\mathcal{A}$  be a category that has finite products.*

*Then  $\mathcal{A}$  is preadditive if and only if it is enriched over the category **AbMon** of abelian monoids, i. e. if all morphism sets of  $\mathcal{A}$  have an abelian monoid structure s. t. composition is bilinear.*

*Proof.* A preadditive category has finite products by definition and Remark B.3 means that it is also enriched over **AbMon**.

Now let  $\mathcal{A}$  be a category that has finite products and is enriched over **AbMon**. For  $X, Y \in \text{ob } \mathcal{A}$  let the “addition” in  $\text{Hom}_{\mathcal{A}}(X, Y)$  be denoted by  $+_{X,Y}$  and its unit by  $0_{X,Y}$ .

$\mathcal{A}$  has in particular a terminal object  $\ast$ . The monoid structure on  $\text{Hom}_{\mathcal{A}}(\ast, \ast)$  is trivial since  $\ast$  is a terminal object and any monoid with only one element is trivial. In particular, we have  $\text{id}_{\ast} = 0_{\ast, \ast}$ .

For all  $X \in \text{ob } \mathcal{A}$ ,  $\text{Hom}_{\mathcal{A}}(\ast, X)$  has a monoid structure, hence is not empty. Now for any  $f: \ast \rightarrow X$  we have  $f = f \circ \text{id}_{\ast} = f \circ 0_{\ast, \ast} = 0_{\ast, X}$  by the bilinearity of composition. Hence  $\text{Hom}_{\mathcal{A}}(\ast, X) = \{0_{\ast, X}\}$  for all  $X \in \text{ob } \mathcal{A}$ , i. e.  $\ast$  is also an initial object and therefore  $\mathcal{A}$  is pointed. From now on  $0$  will denote a zero object in  $\mathcal{A}$ . Note that for any  $X, Y \in \text{ob } \mathcal{A}$ ,  $0_{X,Y}$  is the unique morphism that factors through  $0$ .

Let  $X, Y \in \text{ob } \mathcal{A}$ . We want to endow  $X \times Y$  with the structure of a coproduct of  $X$  and  $Y$  s. t.  $(\text{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \text{id}_Y) = \text{id}_{X \times Y}$ . This enforces the structure morphisms of the coproduct to be  $\text{in}_1 := \text{id}_X \times 0_{X,Y}$  and  $\text{in}_2 := 0_{Y,X} \times \text{id}_Y$ .

Given  $Z \in \text{ob } \mathcal{A}$  and morphisms  $f_1: X \rightarrow Z$  and  $f_2: Y \rightarrow Z$ , define  $f_1 \amalg f_2$  to be  $(f_1 \circ \text{pr}_1) +_{X \times Y, Z} (f_2 \circ \text{pr}_2)$ . Then we have indeed

$$\begin{aligned} & ((f_1 \circ \text{pr}_1) +_{X \times Y, Z} (f_2 \circ \text{pr}_2)) \circ \text{in}_1 = \\ & ((f_1 \circ \text{pr}_1) +_{X \times Y, Z} (f_2 \circ \text{pr}_2)) \circ (\text{id}_X \times 0_{X,Y}) = \\ & ((f_1 \circ \text{pr}_1) \circ (\text{id}_X \times 0_{X,Y})) +_{X,Z} ((f_2 \circ \text{pr}_2) \circ (\text{id}_X \times 0_{X,Y})) = \\ & (f_1 \circ \text{id}_X) +_{X,Z} (f_2 \circ 0_{X,Y}) = \\ & f_1 +_{X,Z} 0_{X,Z} = f_1 \end{aligned}$$

and similarly

$$\begin{aligned} & ((f_1 \circ \text{pr}_1) +_{X \times Y, Z} (f_2 \circ \text{pr}_2)) \circ \text{in}_2 = \\ & ((f_1 \circ \text{pr}_1) \circ (0_{Y,X} \times \text{id}_Y)) +_{Y,Z} ((f_2 \circ \text{pr}_2) \circ (0_{Y,X} \times \text{id}_Y)) = \\ & 0_{Y,Z} +_{Y,Z} f_2 = f_2. \end{aligned}$$

Now let  $f': X \times Y \rightarrow Z$  be another morphism s. t.  $f' \circ \text{in}_1 = f_1$  and  $f' \circ \text{in}_2 = f_2$ .

We first claim that

$$\text{id}_{X \times Y} = (\text{pr}_1 \times 0_{X \times Y, Y}) +_{X \times Y, X \times Y} (0_{X \times Y, X} \times \text{pr}_2).$$

Indeed, by the bilinearity of composition we have

$$\begin{aligned} \text{pr}_1 \circ ((\text{pr}_1 \times 0_{X \times Y, Y}) +_{X \times Y, X \times Y} (0_{X \times Y, X} \times \text{pr}_2)) &= \\ (\text{pr}_1 \circ (\text{pr}_1 \times 0_{X \times Y, Y})) +_{X \times Y, X} (\text{pr}_1 \circ (0_{X \times Y, X} \times \text{pr}_2)) &= \\ \text{pr}_1 +_{X \times Y, X} 0_{X \times Y, X} &= \text{pr}_1 \end{aligned}$$

and similarly

$$\text{pr}_2 \circ ((\text{pr}_1 \times 0_{X \times Y, Y}) +_{X \times Y, X \times Y} (0_{X \times Y, X} \times \text{pr}_2)) = \text{pr}_2.$$

Hence the two morphisms coincide since they agree on both factors.

Moreover, note that we have

$$\text{pr}_1 \circ (\text{id}_X \times 0_{X, Y}) \circ \text{pr}_1 = \text{pr}_1$$

and

$$\text{pr}_2 \circ (\text{id}_X \times 0_{X, Y}) \circ \text{pr}_1 = 0_{X, Y} \circ \text{pr}_1 = 0_{X \times Y, Y},$$

which means that  $\text{in}_1 \circ \text{pr}_1 = (\text{id}_X \times 0_{X, Y}) \circ \text{pr}_1 = \text{pr}_1 \times 0_{X \times Y, Y}$  since these morphisms agree on both factors. Similarly, we also have  $\text{in}_2 \circ \text{pr}_2 = (0_{Y, X} \times \text{id}_Y) \circ \text{pr}_2 = 0_{X \times Y, X} \times \text{pr}_2$ .

Again using the bilinearity of composition, these yield

$$\begin{aligned} f' \circ \text{id}_{X \times Y} &= f' \circ ((\text{pr}_1 \times 0_{X \times Y, Y}) +_{X \times Y, X \times Y} (0_{X \times Y, X} \times \text{pr}_2)) \\ &= (f' \circ (\text{pr}_1 \times 0_{X \times Y, Y})) +_{X \times Y, X \times Y} (f' \circ (0_{X \times Y, X} \times \text{pr}_2)) \\ &= (f' \circ \text{in}_1 \circ \text{pr}_1) +_{X \times Y, X \times Y} (f' \circ \text{in}_2 \circ \text{pr}_2) \\ &= (f_1 \circ \text{pr}_1) +_{X \times Y, X \times Y} (f_2 \circ \text{pr}_2) = f_1 \amalg f_2. \end{aligned}$$

Hence  $\text{in}_1$  and  $\text{in}_2$  do endow  $X \times Y$  with a suitable coproduct structure.  $\square$

Using this characterization we obtain a generic class of preadditive categories.

**Proposition B.5.** *Let  $\mathcal{C}$  be a category which has finite products.*

*Then the category  $\mathcal{C}\text{-AbMon}$  of abelian monoid objects (with homomorphisms of monoid objects between them) is a preadditive category.*

*Proof.* First we note that  $\mathcal{C}\text{-AbMon}$  can alternatively be described as follows:

$$\text{ob}(\mathcal{C}\text{-AbMon}) = \{M \in \text{ob } \mathcal{C} \mid \text{Hom}_{\mathcal{C}}(\_, M) \text{ factors through } \mathbf{AbMon}\},$$

and

$$\begin{aligned} \text{Hom}_{\mathcal{C}\text{-AbMon}}(M, N) = \{f \in \text{Hom}_{\mathcal{C}}(M, N) \mid \\ (f_*)_X: \text{Hom}_{\mathcal{C}}(X, M) \rightarrow \text{Hom}_{\mathcal{C}}(X, N) \\ \text{is a morphisms of (abelian) monoids for all} \\ X \in \text{ob } \mathcal{C}\} \end{aligned}$$

for any  $M, N \in \text{ob}(\mathcal{C}\text{-AbMon})$ .

Hence, for any  $M, N \in \text{ob}(\mathcal{C}\text{-AbMon})$ ,  $\text{Hom}_{\mathcal{C}\text{-AbMon}}(M, N)$  has an abelian monoid structure given by pointwise addition and units on the level of represented functors. Then composition is bilinear w.r.t. this addition since morphisms between monoid objects are chosen to preserve the addition on the homomorphism sets.

This means that  $\mathcal{C}\text{-AbMon}$  enriched over  $\mathbf{AbMon}$  and hence a preadditive category by the previous proposition.  $\square$

**Remark B.6.** Remark B.3 implies also that any object  $X$  of a preadditive category has the structure of an abelian monoid object given by the codiagonal morphism  $\nabla := \begin{pmatrix} 1 & 1 \end{pmatrix}: X \oplus X \rightarrow X$  and the “unit”  $0 \rightarrow X$ . Dually,  $X$  has also the structure of a cocommutative comonoid object given by the diagonal morphism  $\Delta := \begin{pmatrix} 1 \\ 1 \end{pmatrix}: X \oplus X \rightarrow X$  and the “counit”  $X \rightarrow 0$ .

In fact, for a preadditive category  $\mathcal{A}$ , the functor  $\mathcal{A} \rightarrow \mathcal{A}\text{-AbMon}$  which endows an object with the above abelian monoid structure is an equivalence of categories. However, we will neither use nor prove this statement.

As the prefix “pre” suggests, preadditive categories are not quite what we are looking for. We get to the concept of additive categories by requiring that additive inverses of morphisms exist.

**Proposition B.7.** *Let  $\mathcal{A}$  be a preadditive category.*

*Then the following are equivalent:*

(i) *For any  $X \in \text{ob } \mathcal{A}$ , the “shear morphism”*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}: X \oplus X \rightarrow X \oplus X$$

*is an isomorphism.*

(ii) *For any  $X \in \text{ob } \mathcal{A}$ , the identity morphism  $\text{id}_X$  has an additive inverse in  $\text{End}_{\mathcal{A}}(X)$ .*

(iii) *For any  $X, Y \in \text{ob } \mathcal{A}$ , each  $f \in \text{Hom}_{\mathcal{A}}(X, Y)$  has an additive inverse.*

(iv) *For any  $X \in \text{ob } \mathcal{A}$ , the abelian monoid object  $(X, \nabla, 0 \rightarrow X)$  is an (abelian) group object.*

(v) For any  $X \in \text{ob } \mathcal{A}$ , the cocommutative comonoid object  $(X, \Delta, X \rightarrow 0)$  is a (cocommutative) cogroup object.

*Proof.* “(i)  $\Rightarrow$  (ii)”: Let the inverse of the shear morphism of  $X$  be given by

$$\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} : X \oplus X \rightarrow X \oplus X.$$

Then we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} = \begin{pmatrix} j_{1,1} + j_{2,1} & j_{1,2} + j_{2,2} \\ j_{2,1} & j_{2,2} \end{pmatrix}$$

Hence  $j_{1,2} = 0_{X,X}$  and  $j_{1,1} = j_{2,2} = \text{id}_X$ . This yields

$$\text{id}_X + j_{1,2} = j_{2,2} + j_{1,2} = 0_{X,X},$$

so  $j_{1,2}$  is an additive inverse of  $\text{id}_X$ .

“(ii)  $\Rightarrow$  (iii)”: Let  $-\text{id}_X$  be an additive inverse for  $\text{id}_X$ . Then the bilinearity of composition yields

$$f + f \circ (-\text{id}_X) = f \circ \text{id}_X + f \circ (-\text{id}_X) = f \circ (\text{id}_X + (-\text{id}_X)) = f \circ 0_{X,X} = 0_{X,Y},$$

i. e.  $f \circ (-\text{id}_X)$  is an additive inverse for  $f$ .

“(iii)  $\Rightarrow$  (iv)”: Note that  $X$  is a group object in  $\mathcal{A}$  if and only if its represented functor  $\text{Hom}_{\mathcal{A}}(\_, X)$  factors through the category **Grp** of groups. Since  $X$  is an abelian monoid object, we already know that  $\text{Hom}_{\mathcal{A}}(\_, X)$  factors through **AbMon**. Now the fact that for each  $Y \in \text{ob } \mathcal{A}$  each  $f \in \text{Hom}_{\mathcal{A}}(Y, X)$  has an additive inverse implies that the abelian monoids  $(\text{Hom}_{\mathcal{A}}(Y, X), +_{Y,X}, 0_{Y,X})$  are in fact abelian groups. Since all monoid homomorphisms between groups are already homomorphisms of groups, this means that  $\text{Hom}_{\mathcal{A}}(\_, X)$  factors through the category of (abelian) groups.

“(iv)  $\Rightarrow$  (v)”: If  $X$  is a group object with the “multiplication” given by  $\nabla$ , there exists a morphism  $j: X \rightarrow X$  s. t.

$$0_{X,X} = (1 \ 1) \begin{pmatrix} 1 \\ j \end{pmatrix} = \text{id}_X \circ \text{id}_X + \text{id}_X \circ j = \text{id}_X + j.$$

Hence for comonoid structure on  $X$  we obtain

$$(1 \ j) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{id}_X \circ \text{id}_X + j \circ \text{id}_X = \text{id}_X + j = 0_{X,X}.$$

A similar argument shows that the fact that  $j$  is also a “left inverse for the multiplication of  $X$ ” implies that  $j$  is also a “left inverse for the comultiplication of  $X$ ”. In total, we obtain that  $(X, \Delta, X \rightarrow 0, j)$  is a cogroup object.

“(v)  $\Rightarrow$  (i)”: Let  $j: X \rightarrow X$  be the “coinverse” morphism w. r. t.  $\Delta$ . Then calculations similar to the ones in the proof of the previous implication yield that  $\text{id}_X + j = 0_{X,X} = j + \text{id}_X$ . Hence we obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & j+1 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+j \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since  $\text{id}_{X \oplus X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , we see that the shear morphism is an isomorphism.  $\square$

**Definition B.8.** A preadditive category is called *additive* if it satisfies one (and hence all) of the conditions in the previous proposition.

Additive categories also have an alternative characterization similar to the one in Proposition B.4 for preadditive categories.

**Corollary B.9.** *A category  $\mathcal{A}$  which has finite products is additive if and only if it is enriched over the category  $\mathbf{Ab}$  of abelian groups.*

*Proof.* If  $\mathcal{A}$  is additive then the condition (iii) in Proposition B.7 means that  $\mathcal{A}$  is enriched not only over  $\mathbf{AbMon}$ , but even over  $\mathbf{Ab}$  since additive inverses exist in homomorphism monoids and (bi)linear maps of abelian groups are exactly (bi)linear maps of underlying monoids.

Conversely, if  $\mathcal{A}$  is enriched over  $\mathbf{Ab}$ , then  $\mathcal{A}$  is preadditive by Proposition B.4 and the condition (iii) in Proposition B.7 is fulfilled since additive inverses in all homomorphism monoids exist.  $\square$

This characterization yields a generic class of examples which is used in the proof of Corollary 3.2.

**Corollary B.10.** *Let  $\mathcal{C}$  be a category which has finite products.*

*Then the category  $\mathcal{C}\text{-Ab}$  of abelian group objects (with homomorphisms of group objects between them) is an additive category.*

*Proof.* Just as the previous corollary, this follows immediately from Proposition B.4 and the condition (iii) in Proposition B.7.  $\square$

Lastly, let us note that as in the case of preadditive categories, an additive category  $\mathcal{A}$  is in fact equivalent to the category of abelian group objects in  $\mathcal{A}$ .



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