Loop Objects in Pointed Derivators

Aras Ergus

Geboren am 19. Mai 1993 in Osmangazi, Türkei

When?

Bachelorarbeit Mathematik

Betreuer: Dr. Moritz Groth

Zweitgutachter: Prof. Dr. Stefan Schwede

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Zusammenfassung

Derivatore sind abstrakte Mittel, mit denen man Homotopietheorie betreiben kann. Insbesondere können viele Aussagen aus der (klassischen) Homotopietheorie für gewisse Arten von Derivatoren formuliert und bewiesen werden. In dieser Arbeit geht es um eine solche Aussage, nämlich eine Derivatorversion der Tatsache, dass Schleifenräume in der Homotopiekategorie der topologischen Räume eine kanonische Gruppenstruktur besitzen.

Contents

Introduction

Motivation

Derivators provide an abstract framework for homotopy theory. In particular, many statements from (classical) homotopy theory can be formulated and proven for certain kinds of derivators. This thesis is about one such statement, namely a “derivator version” of the fact that the loop spaces have a canonical group object structure in the homotopy category of topological spaces.

About This Thesis

The thesis consists of three regular sections and two appendices. In the first section I deal with the main topic of this thesis, namely the fact that loop objects in values of a pointed derivator are group objects. The main reference for this section is [1]. The second section covers the case of double loop objects and depicts an Eckmann–Hilton argument showing that double loop objects in values of a pointed derivator are indeed abelian group objects. It is followed by a very short section containing a few applications in the general theory of derivators. The first appendix is dedicated to the Segal condition which is used to decide if a given simplicial object is a category object. In the second appendix I try to give a clear and precise description of preadditive and additive categories. It is not much more than an elaboration of Subsection 2.1 of [2].

I omitted a general introduction to the theory of derivators, partially because this thesis would be much longer if it introduced every non-trival concept or statement it used and also because there are a few rather elementary introductory texts about this topic (e. g. [2]) which are more detailed than what I could write for this thesis. However, it would be convenient for the reader to get familiar with derivators before reading this thesis.

Even though the introduction is written from a first person perspective, the “mathematical we” will accompany the reader in the main part of the thesis.

Acknowledgments and Thanks

1 Loop Objects

In this section we prove our main result, i. e. show that loop objects in values of pointed derivators are group objects.

Let $\mathscr{D}$ be a pointed derivator throughout this section.

Simplicial Objects which Induce Loop Objects

The crucial point of our discussion of the loop objects is the fact that they are induced by certain families of objects which fulfill slightly more general conditions than that for a simplicial object.

Notation 1.1.
- Let $\langle n \rangle := \{0, \dots, n\}$ for $n \in \mathbb{N}$. We will consider these as objects of the category **Fin** of finite sets or (equivalently) finite discrete categories.
- Let $\_^{\triangleright} \colon \mathbf{Fin} \to \mathbf{Cat}$ be the cone functor, i. e. the functor which adds a terminal object ∞ to a given category. Let $\lrcorner_n := \langle n \rangle^{\triangleright}$.

Definition 1.2. For $a \in \mathbf{Fin}$ let ω_a be the composition

$$\omega_a \colon \mathscr{D}(*) \xrightarrow{\infty_!} \mathscr{D}(a^{\triangleright}) \xrightarrow{\lim_{a^{\triangleright}}} \mathscr{D}(*).$$

For $n \in \mathbb{N}$ we will abuse notation and write ω_n for $\omega_{\langle n \rangle}$. In particular, we have $\Omega \cong \omega_1 = \lim_{\lrcorner_1} \circ \infty_! \colon \mathscr{D}(*) \to \mathscr{D}(\lrcorner_1) \to \mathscr{D}(*)$ for the loop functor.

Lemma 1.3. *The assignment $a \mapsto \omega_a$ can be made into a "functor"*

$$\omega \colon \mathbf{Fin}^{\mathrm{op}} \to \mathrm{END}_{\mathbf{CAT}}(\mathscr{D}(*)),$$

i. e. for each $a \in \operatorname{ob} \mathbf{Fin}$, ω_a is an endofunctor of $\mathscr{D}(*)$ and we can assign each map $f \colon a \to b$ between finite sets to a natural transformation $\omega_f \colon \omega_a \Rightarrow \omega_b$ so that this assignment is compatible with compositions and identities.

Proof. For $a \in \operatorname{ob} \mathbf{Fin}$, ω_a is an endofunctor of $\mathscr{D}(*)$ by construction.

For functoriality, we consider $a, b \in \operatorname{ob} \mathbf{Fin}$ and $f \colon a \to b$. Then we have two diagrams

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle\infty}\downarrow & & \downarrow{\scriptstyle\infty} \\ a^{\triangleright} & \xrightarrow{f^{\triangleright}} & b^{\triangleright} \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathscr{D}(*) & \longleftarrow & \mathscr{D}(*) \\ {\scriptstyle\infty_!}\downarrow & \searrow & \downarrow{\scriptstyle\infty_!} \\ \mathscr{D}(a^{\triangleright}) & \xleftarrow[(f^{\triangleright})^*]{} & \mathscr{D}(b^{\triangleright}) \\ {\scriptstyle\lim_{a^{\triangleright}}}\downarrow & \nwarrow & \downarrow{\scriptstyle\lim_{b^{\triangleright}}} \\ \mathscr{D}(*) & \longleftarrow & \mathscr{D}(*) \end{array}, \qquad (1)$$

where the second one is obtained from the first by applying $\mathscr{D}$ and then using the appropriate mates.

Now we want to show that the upper natural transformation on the right is an isomorphism and then define the natural transformation $\omega_f : \omega_b \Rightarrow \omega_a$ as the pasting of the two squares on the right.

Note that we can detect such isomorphisms pointwise. In order to do that, we consider an $x \in \mathrm{ob}(a)$, which yields a diagram

$$\begin{array}{ccccc} (\infty/x) & \xrightarrow{\pi} & * & \longrightarrow & * \\ {\scriptstyle\pi}\downarrow & \swarrow & \downarrow{\scriptstyle\infty} & /\!/ & \downarrow{\scriptstyle\infty} \\ * & \xrightarrow[x]{} & a^{\triangleright} & \xrightarrow[f^{\triangleright}]{} & b^{\triangleright} \end{array} .$$

Then we know that the mate transformation $\pi_! \pi^* \Rightarrow x^* \infty_!$ is an isomorphism since the square on the left is a slice square and hence homotopy exact. Futhermore, we have

$$(\infty/x) \cong \begin{cases} \varnothing & x \neq \infty \\ * & x = \infty \end{cases} .$$

Since $f^{\triangleright}(x) = \infty$ iff $x = \infty$, this yields that the pasting of the two squares is a also slice square, hence homotopy exact, which means that the mate transformation $\pi_! \pi^* \Rightarrow (f^{\triangleright}(x))^* \infty_!$ is an isomorphism. Hence, in total, we obtain that the mate transformation $x^* \infty_! \Rightarrow (f^{\triangleright}(x))^* \infty_!$ is an isomorphism.

We can now define $\omega_f : \omega_b \Rightarrow \omega_a$ to be the pasting of the inverse of $\infty_! \Rightarrow (f^{\triangleright})^* \infty_!$ with $\lim_{b^{\triangleright}} \Rightarrow \lim_{a^{\triangleright}} (f^{\triangleright})^*$. This construction is compatible with composition of maps since mates are compatible with pastings. Furthermore, identities are mapped to identites since all the natural transformations in (1) are identities if f is an identity map. □

Corollary 1.4. *For $X \in \mathrm{ob}(\mathscr{D}(*))$, $(\omega_n X)_{n \in \mathbb{N}}$ can be viewed as a simplicial object since the simplex category* $\boldsymbol{\Delta}$ *is a subcategory of* **Fin** *(which is not full).*

Loop Objects as Monoid Objects

Our next step is showing that the simplicial objects associated with loop objects are trivial in the zeroth level and satisfy the Segal condition, which means that loop objects are monoid objects.

Remark 1.5. For $X \in \mathrm{ob}(\mathscr{D}(*))$ we have $\omega_0 X \cong 0^* \infty_! X$ since 0 is the terminal object of $\langle 0 \rangle$, and hence $\omega_0 \cong 0$ since $\infty : * \to \langle 0 \rangle$ is a cosieve.

Proposition 1.6. *Let $n > 1$. We define $i_n : \langle n-1 \rangle \to \langle n \rangle$ to be the inclusion and $i'_n : \langle 1 \rangle \to \langle n \rangle$ to be the function with $i'_n(0) = n-1$ resp. $i'_n(1) = n$.*

Then the natural transformation $\alpha_n : \omega_n \Rightarrow \omega_{n-1} \times \omega_1$ induced by the functor $k_n := i_n^{\triangleright} \amalg i_n'^{\triangleright} : \lrcorner_{n-1} \amalg \lrcorner_1 \to \lrcorner_n$ is an isomorphism.

Proof. Let J_n be the category which is obtained from $\lrcorner_n$ by adding two objects w_0, w_1 with morphisms $w_0 \to k$ for $0 \leqslant k \leqslant n-1$ resp. $w_1 \to k$ for $n-1 \leqslant k \leqslant n$ (and resulting compositions), and let $j_n \colon \lrcorner_n \to J_n$ denote its inclusion functor.

Let $\lrcorner$ be the full subcategory of J_n containing w_0, w_1 and $n-1$ (which is isomorphic to $\lrcorner_1$), and let l_n denote its inclusion functor. Since $n-1$ is terminal in $\lrcorner$, we will denote it also by ∞. Note that l_n has a right adjoint r_n given by

$$r_n(x) = \begin{cases} w_0 & x \in \{w_0, 0, \ldots, n-2\} \\ w_1 & x \in \{w_1, n\} \\ n-1 & x \in \{n-1, \infty\} \end{cases} .$$

for $x \in \operatorname{ob} J_n$, which defines the images of morphisms uniquely. Hence we have $l_n^* \cong (r_n)_*$.

Then, using the natural equivalence $\mathscr{D}(A \amalg B) \simeq \mathscr{D}(A) \times \mathscr{D}(B)$ for $A, B \in \operatorname{ob} \mathbf{Cat}$ and appropriate mates, we obtain a diagram

$$\begin{array}{ccccc}
\mathscr{D}(\ast) & \xrightarrow{(\mathrm{id} \amalg \mathrm{id})^*} & \mathscr{D}(\ast \amalg \ast) & & \\
{\scriptstyle (\infty_n)_!}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle (\infty_{n-1} \amalg \infty_1)_!} & & \\
\mathscr{D}(\lrcorner_n) & \xrightarrow{k_n^*} & \mathscr{D}(\lrcorner_{n-1} \amalg \lrcorner_1) & & \\
{\scriptstyle (j_n)_*}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle (\pi_{n-1} \amalg \pi_1)_*} & & \\
\mathscr{D}(J_n) & \xrightarrow{(w_0 \amalg w_1)^*} & \mathscr{D}(\ast \amalg \ast) & \xrightarrow{(\pi_{\ast \amalg \ast})_*} & \mathscr{D}(\ast) \\
{\scriptstyle l_n^* \cong (r_n)_*}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle (w_0 \amalg w_1)_*} \ \cong & & \big\downarrow{\scriptstyle \mathrm{id}_*} \\
\mathscr{D}(\lrcorner) & \xrightarrow{\mathrm{id}^*} & \mathscr{D}(\lrcorner) & \xrightarrow{(\pi_1)_*} & \mathscr{D}(\ast)
\end{array} .$$

Under the equivalences mentioned above the upper natural transformation is given by

$$(\infty_{n-1})_! \times (\infty_1)_! \Rightarrow \left((i_n^{\triangleright})^* \times (i_n'^{\triangleright})^* \right) (\infty_n)_!,$$

which is the product of the natural transformations which occur in the definition of ω_{i_n} resp. $\omega_{i'_n}$ (see (1) in the proof of Lemma 1.3). Hence it is an isomorphism as the product of two natural isomorphisms. Furthermore, the right square in the last row commutes up to isomorphism since $\pi_1(w_0 \amalg w_1) = \pi_{\ast \amalg \ast}$.

All in all, the diagram above yields a natural transformation from

$$(\pi_1)_*(r_n)_*(j_n)_*(\infty_n)_! \cong (\pi_n)_*(\infty_n)_! \cong \omega_n$$

to

$$(\pi_{\ast \amalg \ast})_*(\pi_{n-1} \amalg \pi_1)_*(\infty_{n-1} \amalg \infty_1)_! \cong ((\pi_{n-1})_* \infty_{n-1}) \times ((\pi_1)_* \infty_1) \cong \omega_{n-1} \times \omega_1$$

which is the α_n mentioned in the statement of this proposition. We now want to show that (certain restrinctions of) the natural transformations in the remaining two squares are isomorphisms, which will imply that α_n is an isomorphism.

For the middle square we consider diagrams of the form

$$\begin{array}{ccccc} (x/(\pi_{n-1} \amalg \pi_1)) & \xrightarrow{p_{x,\pi_{n-1}\amalg\pi_1}} & \lrcorner_{n-1} \amalg \lrcorner_1 & \xrightarrow{k_n} & \lrcorner_n \\ {\scriptstyle \pi_{(x/(\pi_{n-1}\amalg\pi_1))}}\downarrow & \Rightarrow & {\scriptstyle \pi_{n-1}\amalg\pi_1}\downarrow & \Rightarrow & \downarrow{\scriptstyle j_n} \\ \ast & \xrightarrow[x]{} & \ast \amalg \ast & \xrightarrow[w_0 \amalg w_1]{} & J_n \end{array}$$

for $x \in \mathrm{ob}(\ast \amalg \ast) = \{*_0, *_1\}$.

Then we have

$$(*_0/(\pi_{n-1} \amalg \pi_1)) \cong \lrcorner_{n-1} \quad \text{and} \quad (*_1/(\pi_{n-1} \amalg \pi_1)) \cong \lrcorner_1$$

where under this identification $p_{*_0,\pi_{n-1}\amalg\pi_1}$ resp. $p_{*_1,\pi_{n-1}\amalg\pi_1}$ is given by the inclusion ι_0 resp. ι_1 of the corresponding category. Since the left square is a slice square, this means that

$$*_0^*(\pi_{n-1} \amalg \pi_1)_* \Rightarrow (\pi_{(*_0/(\pi_{n-1}\amalg\pi_1))})_*(p_{*_0,\pi_{n-1}\amalg\pi_1})^* \cong (\pi_{n-1})_*\iota_0^*$$

and

$$*_1^*(\pi_{n-1} \amalg \pi_1)_* \Rightarrow (\pi_{(*_1/(\pi_{n-1}\amalg\pi_1))})_*(p_{*_1,\pi_{n-1}\amalg\pi_1})^* \cong (\pi_1)_*\iota_1^*$$

are isomorphisms.

On the other hand, we also have

$$(w_0/j_n) \cong \lrcorner_{n-1} \quad \text{and} \quad (w_1/j_n) \cong \lrcorner_1,$$

where under this identification p_{w_0,j_n} is given by $i_n^{\triangleright} = k_n\iota_0$ and p_{w_1,j_n} is given by $i_n'^{\triangleright} = k_n\iota_1$. Hence the pasting of the above squares is (up to isomorphisms) also a slice square, so the natural transformations

$$((w_0 \amalg w_1)*_0)^*(j_n)_* = w_0^*(j_n)_* \Rightarrow (\pi_{(w_0/j_n)})_*(p_{w_0,j_n})^* \cong (\pi_{n-1})_*(i_n^{\triangleright})^*$$

and

$$((w_0 \amalg w_1)*_1)^*(j_n)_* = w_1^*(j_n)_* \Rightarrow (\pi_{(w_1/j_n)})_*(p_{w_1,j_n})^* \cong (\pi_1)_*(i_n^{\triangleright})^*$$

are also isomorphisms.

Combining these isomorphisms, we see that

$$\begin{aligned} *_0^*(\pi_{n-1} \amalg \pi_1)_*k_n^* &\cong (\pi_{n-1})_*\iota_0^*k_n^* \\ &\cong (\pi_{n-1})_*(i_n^{\triangleright})^* \cong w_0^*(j_n)_* \\ &\cong ((w_0 \amalg w_1)*_0)^*(j_n)_* \cong *_0^*(w_0 \amalg w_1)^*(j_n)_* \end{aligned}$$

and

$$\begin{aligned} *_1^*(\pi_{n-1} \amalg \pi_1)_* k_n^* &\cong (\pi_{n-1})_* \iota_1^* k_n^* \\ &\cong (\pi_{n-1})_* (i_n'^{\triangleright})^* \cong w_1^* (j_n)_* \\ &\cong ((w_0 \amalg w_1) *_1)^* (j_n)_* \cong *_1^* (w_0 \amalg w_1)^* (j_n)_*. \end{aligned}$$

Since mates are compatible with pastings this means that the natural transformation $x^*(w_0 \amalg w_1)^*(j_n)_* \Rightarrow x^*(\pi_{n-1} \amalg \pi_1)_* k_n^*$ is an isomorphism for all $x \in \mathrm{ob}(\ast \amalg \ast)$, hence it is an isomorphism as isomorphisms can be detected pointwise.

Note that, in general, the natural transformation

$$l_n^* \cong (r_n)_* \Rightarrow (w_0 \amalg w_1)_* (w_0 \amalg w_1)^*$$

in the first square of the last row is not an isomorphism for all $X \in \mathscr{D}(J_n)$. We are going to "fix" this by restricting our attention to $\mathrm{essim}((j_n)_*(\infty_n)_!)$.

First, we compute $(n-1)^*(j_n)_* X'$ for $X' \in \mathrm{essim}(\infty_!)$: Consider the slice square

$$\begin{array}{ccc} (n-1/j_n) & \xrightarrow{p} & \lrcorner_n \\ {\scriptstyle \pi}\downarrow & \nearrow & \downarrow{\scriptstyle j_n} \\ \ast & \xrightarrow[n-1]{} & J_n \end{array} .$$

Then we know that $(n-1)^*(j_n)_* \Rightarrow \pi_* p^*$ is an isomorphism.

Now $(n-1/j_n)$ is isomorphic to the full subcategory K_n of $\lrcorner_n$ spanned by $n-1$ and ∞, where p corresponds to the inclusion $K_n \to \lrcorner_n$ under this identification. Hence we see that $\pi_* p^* \cong (n-1)^* p^* \cong (p(n-1))^*$ since $n-1$ is the initial object of K_n. Therefore $(n-1)^*(j_n)_* \cong (n-1)^*$, where the former $n-1$ is the object in J_n and the latter the one in $\lrcorner_n$. Since $\infty \colon \ast \to \lrcorner_n$ is a cosieve we know that $(n-1)^* X' \cong 0$ for $X' \in \mathrm{essim}(\infty_!)$, so we obtain $(n-1)^*(j_n)_* X' \cong 0$.

This means that for $X \in \mathrm{essim}((j_n)_*(\infty_n)_!)$ we have $\infty^* l_n^* X \cong (l_n \infty)^* X \cong (n-1)^* X \cong 0$. On the other hand, for any $Y \in \mathscr{D}(\ast \amalg \ast)$, we have $\infty^*(w_0 \amalg w_1)_* Y \cong 0$ since $w_0 \amalg w_1$ is a sieve. Hence $l_n^* \cong (r_n)_*$ and $(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ agree on $\infty = n-1$ for $X \in \mathrm{essim}((j_n)_*(\infty_n)_!)$.

We now consider w_i for $i \in \{0, 1\}$. In the slice square

$$\begin{array}{ccc} (w_i/w_0 \amalg w_1) & \xrightarrow{p} & \ast \amalg \ast \\ {\scriptstyle \pi}\downarrow & \nearrow & \downarrow{\scriptstyle w_0 \amalg w_1} \\ \ast & \xrightarrow[w_i]{} & \lrcorner \end{array} ,$$

$(w_i/w_0 \amalg w_1)$ can be identified with $\ast$ and p with $*_i \colon \ast \to \ast \amalg \ast$. Hence we see that $w_i^*(w_0 \amalg w_1)_* \Rightarrow \pi_* w_i^* \cong w_i^*$ is an isomorphism. This yields

$$\begin{aligned} w_i^*(w_0 \amalg w_1)_*(w_0 \amalg w_1)^* &\overset{\cong}{\Rightarrow} w_i^*(w_0 \amalg w_1)^* = ((w_0 \amalg w_1) w_i)^* \\ &= w_i^* = (l_n w_i)^* = w_i^* l_n^* \cong w_i^* (r_n)_*, \end{aligned}$$

so $w_i^*(r_n)_* \Rightarrow w_i^*(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ is an isomorphism by the compatibility of mates with pastings.

All in all, $(r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ is an isomorphism pointwise, so it is indeed an isomorphism. This means that the last remaining square is also filled with an isomorphism, so $\alpha_n \colon \omega_n \Rightarrow \omega_{n-1} \times w_1$ is an isomorphism in total. □

Corollary 1.7. *Let $\mu\colon \langle 1\rangle \to \langle 2\rangle$ be the map with $\mu(0) = 0$ resp. $\mu(1) = 2$ and let ε be the unique map from $\langle 1\rangle$ to $\langle 0\rangle$.*

Then for any $X \in \mathrm{ob}(\mathscr{D}(\ast))$, $\omega_1 X \cong \Omega X$ has a monoid object structure given by the multiplication

$$m_X \colon \omega_1 X \times \omega_1 X \xrightarrow{(\alpha_2^{-1})_X} \omega_2 X \xrightarrow{(\omega_\mu)_X} \omega_1 X$$

and the unit

$$0\colon 0 \xrightarrow{\cong} \omega_0 X \xrightarrow{(\omega_\varepsilon)_X} \omega_1 X.$$

Proof. The previous proposition and the preceding remark imply that the Segal morphism

$$\omega_n X \to (\omega_1 X)^n$$

is an isomorphism for any $n \in \mathbb{N}$. Therefore we have $X_0 \cong 0$ and the simplicial set induced by $\omega\_X \colon \mathbf{Fin}^{\mathrm{op}} \to \mathscr{D}(\ast)$ satisfies the Segal condition, so it is a special simplicial object. Hence $\omega_1 X = \Omega X$ has a natural monoid object structure (see Proposition A.4).

Now note that $\langle n\rangle \in \mathrm{ob}\,\mathbf{Fin}$ and $[n] \in \mathrm{ob}\,\mathbf{\Delta}$ are equal as sets for all $n \in \mathbb{N}$. Furthermore, we have $i_2 = \phi^0$, $i_2' = \phi^1$, $\mu = \delta^1$ and $\varepsilon = \sigma^0$ as maps between sets. Hence the monoid object structure on $\omega_1 X$ which is induced by the special simplicial set corresponding to ω_X is indeed given by the morphisms m_X and 0. □

Loop Objects as Group Objects

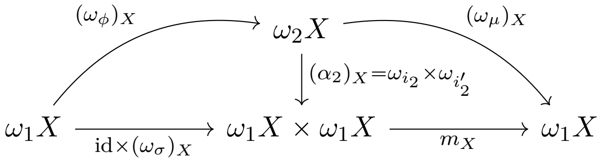
The last step in this section is the construction of inverses for the multiplication of loop objects, concluding that loop objects are group objects.

Proposition 1.8. *Let $\sigma\colon \langle 1\rangle \to \langle 1\rangle$ be the only non-trivial automorphism, i. e. the map swapping 0 and 1.*

Then, for any $X \in \mathscr{D}(\ast)$, there is an inversion morphism for the multiplication of $\Omega X \cong \omega_1 X$ which is given by $(\omega_\sigma)_X \colon \omega_1 X \to \omega_1 X$.

Proof. We have to show that the composition $z := m_X \circ (\mathrm{id}_X \times (w_\sigma)_X)$ factors through $\omega_0 X \cong 0$, i. e. is the zero morphism. In order to do this we will describe z as a morphism which factors through $\omega_2 X$.

Let $\phi\colon \langle 2\rangle \to \langle 1\rangle$ be the map with $\phi(0) = 0 = \phi(2)$ and $\phi(1) = 1$. We claim that the diagram



commutes.

The right triangle commutes by the definition of m_X. We verify the commutativity of the left triangle componentwise. Indeed, we have

$$\begin{aligned}\mathrm{pr}_1 \circ (\omega_{i_2} \times \omega_{i'_2}) \circ (\omega_\phi)_X &= \omega_{i_2} \circ (\omega_\phi)_X = (\omega_{\phi\circ i_2})_X = (\omega_{\mathrm{id}_{\langle 1\rangle}})_X \\ &= \mathrm{id}_{\omega_1 X} = \mathrm{pr}_1 \circ (\mathrm{id}_{\omega_1 X} \times (\omega_\sigma)_X)\end{aligned}$$

and

$$\begin{aligned}\mathrm{pr}_2 \circ (\omega_{i_2} \times \omega_{i'_2}) \circ (\omega_\phi)_X &= \omega_{i'_2} \circ (\omega_\phi)_X = (\omega_{\phi\circ i'_2})_X = (\omega_\sigma)_X \\ &= \mathrm{pr}_2 \circ (\mathrm{id}_{\omega_1 X} \times (\omega_\sigma)_X)\end{aligned}$$

since $\phi \circ i_2 = \mathrm{id}_{\langle 1\rangle}$ and $\phi \circ i'_2 = \sigma$.

Hence we obtain that

$$\begin{aligned} z &= m_X \circ (\mathrm{id}_X \times (w_\sigma)_X) \\ &= (\omega_\mu)_X \circ (\alpha_2^{-1})_X \circ (\alpha_2)_X \circ (\omega_\phi)_X \\ &= (\omega_\mu)_X \circ (\omega_\phi)_X = (\omega_{\phi\circ\mu})_X.\end{aligned}$$

Now note that $\phi \circ \mu$ factors through $\langle 0\rangle$ as $\phi(\mu(0)) = \phi(0) = 0 = \phi(2) = \phi(\mu(0))$. Hence $z = (\omega_{\phi\circ\mu})_X$ factors through $\omega_0 X \cong 0$. $\square$

2 Double Loop Objects

Our next aim is to show that the group object structure on a double loop object is abelian.

In this section $\mathscr{D}$ will again be a pointed derivator.

Loop Functor as a Functor to Group Objects

An important result about the loop functor is the fact that it factors through the category $\mathscr{D}(\ast)$-**Grp** of group objects in $\mathscr{D}(\ast)$ also on the level of morphisms.

Lemma 2.1. *Let $f\colon X \to Y$ be a morphism in $\mathscr{D}(\ast)$.*

Then the induced morphism $\Omega f\colon \Omega X \to \Omega Y$ is a homomorphism of group objects in $\mathscr{D}(\ast)$, where ΩX and ΩY are endowed with the group object structure discussed in the previous section.

Proof. First we note that the functors $\omega\_X, \omega\_Y\colon \mathbf{Fin}^{\mathrm{op}} \to \mathscr{D}(\ast)$ induce special simplicial objects as discussed in the previous section. Furthermore, a morphism $f\colon X \to Y$ induces morphisms $\omega_a f\colon \omega_a X \to \omega_a Y$ for $a \in \operatorname{ob} \mathbf{Fin}$. This assignment is natural in a since for a given $u\colon a \to b$, the diagram

$$\begin{array}{ccc} \omega_a X & \xrightarrow{\omega_a f} & \omega_a Y \\ {\scriptstyle (\omega_u)_X}\uparrow & & \uparrow{\scriptstyle (\omega_u)_Y} \\ \omega_b X & \xrightarrow[\omega_b f]{} & \omega_b Y \end{array}$$

commutes since ω_u is a natural transformation by Lemma 1.3.

Hence $\omega\_f\colon \omega\_X \Rightarrow \omega\_Y$ induces a morphism of monoid objects

$$\Omega X = \omega_1 X \xrightarrow{\omega_1 f = \Omega f} \omega_1 Y = \Omega Y$$

as a natural transformation between special simplicial objects (see Proposition A.6). Now any morphism of monoid objects between group objects is already a morphism of group objects. (This can be, for example, checked on represented functors and hence can be reduced to the fact that a monoid homomorphism between groups is already a group homomorphism.) □

Products under the Loop Functor

The next step in this section is showing that Ω preserves products.

Remark 2.2. Note that the functor $\Omega\colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ has a left adjoint $\Sigma\colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ (see [2, Proposition 8.18]).

Hence Ω preserves limits. In particular, the natural morphism

$$\Omega\left(\prod_{i\in I} X_i\right) \xrightarrow{\prod_{i\in I}\Omega(\mathrm{pr}_i)} \prod_{i\in I} \Omega X_i$$

is an isomorphism for any index set I and any family $(X_i)_{i\in I}$ of objects in $\mathscr{D}(\ast)$.

This immidiately implies that also the group object structure on loop objects are compatible with products.

Remark 2.3. For $X, Y \in \operatorname{ob}\mathscr{D}(\ast)$, the morphism $\Omega(X \times Y) \xrightarrow{\Omega(\mathrm{pr}_1)\times\Omega(\mathrm{pr}_2)} \Omega X \times \Omega Y$ is a homomorphism of group objects since it is a product of group object homomorphisms.

This endows $\Omega(X \times Y)$ with the structure of a product of ΩX and ΩY as group objects s. t.

$$m_{\Omega(X\times Y)} = (\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2))^{-1} \circ \mathrm{mult}_{\Omega X\times\Omega Y} \circ(\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2)),$$

where $\mathrm{mult}_{\Omega X\times\Omega Y} \colon (\Omega X \times \Omega Y) \times (\Omega X \times \Omega Y) \to \Omega X \times \Omega Y$ is the multiplication morphism of the product group object.

Furthermore, the compatibility of Ω with products yields a "new" group object structure on double loop objects.

Corollary 2.4. *For $X \in \operatorname{ob}(\mathscr{D}(\ast))$, $\Omega^2(X)$ has (in addition to the one given by being the loop object of $\Omega(X)$) a group object structure given by the multiplication*

$$m'_X \colon \Omega^2(X) \times \Omega^2(X) \xrightarrow{\cong} \Omega(\Omega X \times \Omega X) \xrightarrow{\Omega(m_X)} \Omega(\Omega(X)) = \Omega^2 X,$$

the unit

$$0 \to \Omega^2 X$$

and inverses

$$\Omega^2 X \xrightarrow{\Omega((\omega_\sigma)_X)} \Omega^2 X.$$

Proof. The commutativity of the required diagrams follow from the fact that the corresponding diagrams commute before applying Ω. ☐

The Eckmann–Hilton Argument

We now have everything at hand to immitate the standard proof of the fact that a group object in **Grp** is an abelian group in order to show that the group object structure on a double loop object is abelian.

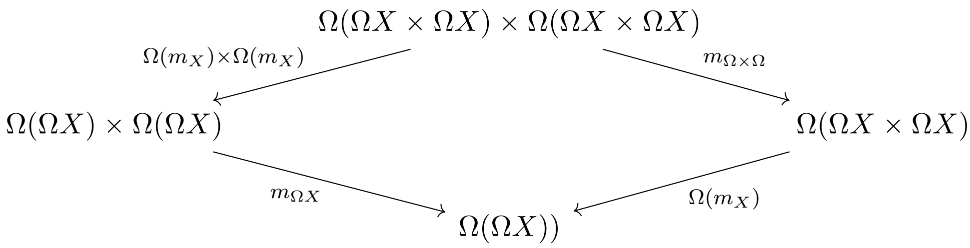
Lemma 2.5. *Let $X \in \operatorname{ob}\mathscr{D}(\ast)$. Let $s_{2,3} := \mathrm{pr}_1 \times \mathrm{pr}_3 \times \mathrm{pr}_2 \times \mathrm{pr}_4 \colon (\Omega^2 X)^4 \to (\Omega^2 X)^4$ be the morphism which "swaps the second and the third factor".*

Then the diagram

$$\begin{array}{ccc}
\Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X & \xrightarrow{s_{2,3}} & \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X \\
\downarrow{\scriptstyle m'_X \times m'_X} & & \downarrow{\scriptstyle m_{\Omega X} \times m_{\Omega X}} \\
\Omega^2 X \times \Omega^2 X & & \Omega^2 X \times \Omega^2 X \\
& {\searrow}^{m_{\Omega X}} \quad \Omega^2 X \quad {\swarrow}^{m'_X} &
\end{array}$$

is commutative.

Proof. We first note that the diagram

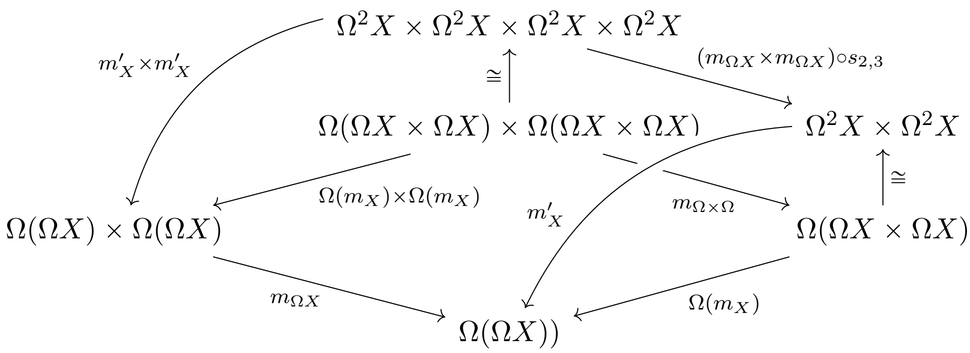


(2)

commutes since $\Omega(m_X)\colon \Omega(\Omega X \times \Omega X) \to \Omega(\Omega(X))$ is a homomorphism of group objects by Lemma 2.1.

Now $(m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3}$ is the multiplication morphism of $\Omega^2 X \times \Omega^2 X$, which also coincides with $(\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2)) \circ m_{\Omega X \times \Omega X} \circ (\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2))^{-1}$ by Remark 2.3.

Hence, identifying $\Omega(\Omega X \times \Omega X)$ with $\Omega^2 X \times \Omega^2 X$, the diagram (2) becomes a commutative diagram



,

which contains the required diagram. ☐

Corollary 2.6. *The "group laws" $m_{\Omega X}$ and m'_X on $\Omega^2 X$ coincide and are abelian.*

In particular, $\Omega^2 \colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ factors through the category $\mathbf{Ab}_{\mathscr{D}(\ast)}$ of abelian group objects in $\mathscr{D}(\ast)$ since each homomorphism of group objects between abelian group objects is a homomorphism of abelian group objects and vice versa.

Proof. Consider the morphism

$$f := \mathrm{pr}_1 \times 0 \times 0 \times \mathrm{pr}_2 \colon \Omega^2 X \times \Omega^2 X \to \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X.$$

Then we have $\mathrm{pr}_1 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f = m_{\Omega_X} \circ (\mathrm{pr}_1 \times 0) = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f = m_{\Omega_X} \circ (\mathrm{pr}_2 \times 0) = \mathrm{pr}_1$ since $0 \to \Omega^2 X$ is the unit morphism for m_{Ω_X}. Hence we have $(m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f = \mathrm{id}_{\Omega^2 X \times \Omega^2 X}$ as these morphisms agree after composing with each of the projections.

Furthermore, we also have $\mathrm{pr}_1 \circ (m'_X \times m'_X) \circ f = m'_X \circ (\mathrm{pr}_1 \times 0) = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (m'_X \times m'_X) \circ f = m'_X \circ (\mathrm{pr}_2 \times 0) = \mathrm{pr}_1$ since $0 \to \Omega^2 X$ is also the unit morphism for m'_X. Hence $(m'_X \times m'_X) \circ f = \mathrm{id}_{\Omega^2 X \times \Omega^2 X}$ as these agree after composing with each of the projections.

In total, using the Eckmann–Hilton identity from the previous lemma, we obtain

$$\begin{aligned} m_{\Omega X} &= m_{\Omega X} \circ \mathrm{id}_{\Omega^2 X \times \Omega^2 X} \\ &= m_{\Omega X} \circ (m'_X \times m'_X) \circ f \\ &= m'_X \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f \\ &= m'_X \circ \mathrm{id}_{\Omega^2 X \times \Omega^2 X} = m'_X. \end{aligned}$$

For the commutativity of $m_{\Omega X} = m'_X$ we consider the morphism

$$g := 0 \times \mathrm{pr}_1 \times \mathrm{pr}_2 \times 0 \colon \Omega^2 X \times \Omega^2 X \to \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X.$$

Then we have $\mathrm{pr}_1 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g = m_{\Omega_X} \circ (0 \times \mathrm{pr}_2) = \mathrm{pr}_2$ and $\mathrm{pr}_2 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g = m_{\Omega_X} \circ (\mathrm{pr}_1 \times 0) = \mathrm{pr}_1$, therefore $(m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g = \mathrm{pr}_2 \times \mathrm{pr}_1$, i. e. the "swapping morphism". On the other hand, we also have $\mathrm{pr}_1 \circ (m'_X \times m'_X) \circ g = m'_X \circ (0 \times \mathrm{pr}_1) = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (m'_X \times m'_X) \circ g = m'_X \circ (\mathrm{pr}_2 \times 0) = \mathrm{pr}_2$, so $(m'_X \times m'_X) \circ g = \mathrm{id}_{\Omega^2 X \times \Omega^2 X}$.

Hence, the Eckmann–Hilton identity yields

$$\begin{aligned} m_{\Omega X} &= m_{\Omega X} \circ \mathrm{id}_{\Omega^2 X \times \Omega^2 X} \\ &= m_{\Omega X} \circ (m'_X \times m'_X) \circ g \\ &= m'_X \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g \\ &= m'_X \circ (\mathrm{pr}_2 \times \mathrm{pr}_1) \\ &= m_{\Omega X} \circ (\mathrm{pr}_2 \times \mathrm{pr}_1), \end{aligned}$$

which means that $m_{\Omega X} = m'_X$ is indeed a commutative multiplication. □

3 Applications

Besides the intristic motivation for studying it, the loop functor can be used to show that stable derivators are additive.

Corollary 3.1. *Let $\mathscr{D}$ be a stable derivator. Then $\mathscr{D}(\ast)$ is an additive category.*

Proof. First of all, $\mathscr{D}(\ast)$ has all products since $\mathscr{D}$ is a derivator. Furthermore, we know that Ω (and hence Ω^2) is an equivalence of categories since $\mathscr{D}$ is stable.

Now note that $\Omega^2 \colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ factors through the category $\mathscr{D}(\ast)$-**Ab** of abelian group objects in $\mathscr{D}(\ast)$ by Corollary 2.6 since morphisms between abelian group objects are simply homomorphisms of underlying group objects.

Hence $\mathscr{D}(\ast) \simeq \mathscr{D}(\ast)$-**Ab**. Now the category of abelian group objects in a category with all finite products is an additive category (see Corollary B.10). □

Furthermore, note that one can construct the "shifted derivator" $\mathscr{D}^A$ for a given small category A, which is given by $\mathscr{D}^A(B) = \mathscr{D}(A \times B)$ on small categories, $(u^*)^{\mathscr{D}^A} = (\mathrm{id}_A \times u)^{\mathscr{D}}$ on functors and $(\gamma^*)^{\mathscr{D}^A} = (\mathrm{id}_{\mathrm{id}_A} \times \gamma)^{\mathscr{D}}$ on natural transformations (see [2, Proposition 7.32]). Then $\mathscr{D}^A$ is pointed resp. stable if $\mathscr{D}$ is so, hence we can obtain statements about $\mathscr{D}(A) \simeq \mathscr{D}^A(\ast)$ by considering $\mathscr{D}^A$ as a derivator.

Remark 3.2. Let A be a small category.

Then the shifted loop functor

$$\Omega^A := (\mathrm{id}_A \times \pi_{\lrcorner})_\ast \circ (\mathrm{id}_A \times \infty)_! \colon \mathscr{D}(A) \to \mathscr{D}(A)$$

factors through $\mathscr{D}$(A)-**Grp** and the double shifted loop functor $\left(\Omega^A\right)^2$ factors even through $\mathscr{D}$(A)-**Ab**.

Moreover, $\mathscr{D}(A)$ is an additive category if $\mathscr{D}$ (and hence $\mathscr{D}^A$) is stable.

A The Segal Condition

In this appendix we will justify Corollary 1.7 by showing that a certain type of simplicial objects give rise to monoid objects.

We start with a review of simplicial objects.

Notation A.1. Let $\mathbf{\Delta}$ be the simplex category, i. e. the category of non-empty finite ordinal numbers. For $n \in \mathbb{N}$ set $[n] = \{0, \dots, n\}$.

For $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$ we fix notation for the following morphisms in $\mathbf{\Delta}$:

- $\delta^{n,i} \colon [n-1] \to [n]$, $n > 0$, is the unique map which "skips i",
- $\sigma^{n,i} \colon [n+1] \to [n]$ is the unique map which "collapses $i+1$ to i",
- $\phi^{n,i} \colon [1] \to [n]$, $i < n$, is the inclusion of $\{i, i+1\}$.

In most cases, we will omit the index n if it is clear from the context.

Given a category $\mathscr{C}$ and a simplicial object $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$, we will denote $X([n])$ by X_n. Then the above maps induce:

- $d_i^X := X(\delta^i) \colon X_n \to X_{n-1}$, the *i-th face map*,
- $s_i^X := X(\sigma^i) \colon X_n \to X_{n+1}$, the *i-th degeneracy map*,
- $f_i^X := X(\phi^i) \colon X_n \to X_1$.

The simplicial object in consideration will mostly be clear from the context and we will omit the upper index X in these cases.

In a certain sense, face maps and degeneracy maps determine a simplicial object uniquely.

Remark A.2. All morphisms in $\mathbf{\Delta}$ can be written as a composition of suitable δ^i's and σ^i's. These maps satisfy the *simplicial relations*:

- $\delta^j \circ \delta^i = \delta^i \circ \delta^{j-1}$ for $i < j$,
- $\sigma^j \circ \delta^i = \delta^i \circ \sigma^{j-1}$ for $i < j$,
- $\sigma^j \circ \delta^i = \mathrm{id}$ for $i = j$ and $i = j+1$,
- $\sigma^j \circ \delta^i = \delta^{i-1} \circ \sigma^j$ for $i > j$,
- $\sigma^j \circ \sigma^i = \sigma^{i-1} \circ \sigma^j$ for $i > j$.

Furthermore, all relations between the δ^i's and the σ^i's are implied by these relations in the following sense:

For a category $\mathscr{C}$, a collection $(X_n)_{n\in\mathbb{N}}$ of objects in $\mathscr{C}$ with morphisms $d_i\colon X_{n-1} \to X_n$ for $n > 0$, $0 \leqslant i \leqslant n$ and $s_i\colon X_n \to X_{n+1}$ for $0 \leqslant i \leqslant n$ yields a simplicial object X s. t. $d_i = X(\delta^i)$ and $s_i = X(\sigma^i)$ iff the *simplicial identities* (which are induced by the simplicial relations) hold:

- $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$,
- $d_i \circ s_j = s_{j-1} \circ d_i$ for $i < j$,
- $d_i \circ s_j = \mathrm{id}$ for $i = j$ and $i = j + 1$,
- $d_i \circ s_j = s_j \circ d_{i-1}$ for $i > j$,
- $s_i \circ s_j = s_j \circ s_{i-1}$ for $i > j$.

A relevant fact in the theory of simplicial sets is that a simplicial set $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to the nerve of a (small) category if and only if the *Segal condition* is satisfied, i. e. for any $n \in \mathbb{N}$, the natural map

$$X_n \xrightarrow{\prod_{i=0}^{n-1} f_i} X_1^{\times_{X_0} n}$$

is an isomorphism. Since (small) categories with only one object can be identified with monoids, this means that simplicial sets X which have exactly one 0-simplex (i. e. $X_0 \cong \{*\}$) and fulfill the Segal condition can also be identified with monoids.

In the following we want to prove a similar statement for simplicial objects in a category. We know that, in general, fiber products don't exist in values of a derivator, but products do. Therefore we will restrict our attention to simplicial objects X with $X_0 \cong \ast$ for a terminal object $\ast$, so that we have $X_1^n \cong X_1^{\times_{X_0} n}$, which makes calculations easier.

In the rest of this appendix $\mathscr{C}$ will be a category which has all finite products (hence an terminal object $\ast$) and $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$ a simplicial object in $\mathscr{C}$.

Definition A.3. X is called *special* if $X_0 \cong \ast$ and X satisfies the Segal condition, i. e. $X_n \xrightarrow{\prod_{i=0}^{n-1} f_i} X_1^n$ is an isomorphism for all $n \in \mathbb{N}$.

We will denote the category of special simplicial objects in $\mathscr{C}$ with natural transformations between those as morphisms by $\mathbf{ss}\mathscr{C}$.

First we show that special simplicial objects give rise to monoid objects.

Proposition A.4. *Let X be a special simplicial object in $\mathscr{C}$.*

Then X_1 has a monoid object structure given by the multiplication morphism

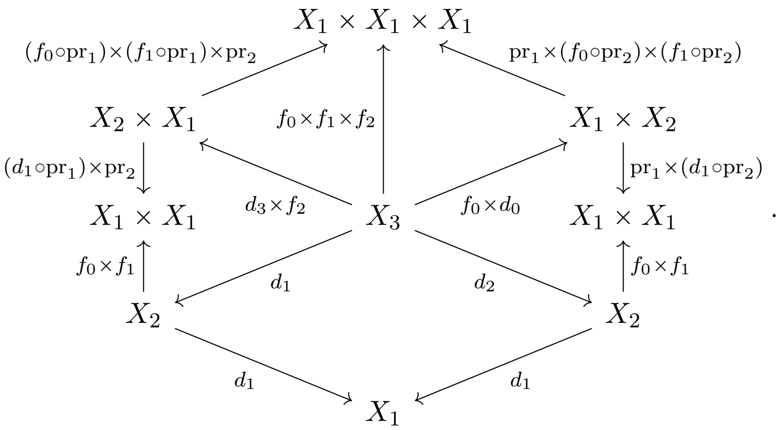
$$m_X\colon X_1 \times X_1 \xrightarrow{(f_0\times f_1)^{-1}} X_2 \xrightarrow{d_1} X_1$$

and the unit morphism

$$e_X\colon * \xrightarrow{\cong} X_0 \xrightarrow{s_0} X_1,$$

where we will omit the index X if it is clear from the context.

Proof. For associativity we consider the diagram



The lower parallelogram commutes as $d_1 \circ d_2 = d_1 \circ d_1$ is one of the simplicial identities. Note that the upper left and upper right sides of the diagram are symmetric, so we will only show that the upper left part is commutative since the commutativity of the other part can be shown similarly.

For the upper left triangle we have

$$\begin{aligned}\mathrm{pr}_1 \circ ((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= (f_0 \circ \mathrm{pr}_1) \circ (d_3 \times f_2)\\ &= f_0 \circ d_3 = f_0\\ &= \mathrm{pr}_1 \circ (f_0 \times f_1 \times f_2)\end{aligned}$$

since $\phi^0 = \delta^3 \circ \phi^0 \colon [1] \to [2] \to [3]$. Similarly, we also have

$$\begin{aligned}\mathrm{pr}_2 \circ ((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= f_1 \circ d_3\\ &= f_1 = \mathrm{pr}_2 \circ (f_0 \times f_1 \times f_2)\end{aligned}$$

since $\phi^1 = \delta^3 \circ \phi^1 \colon [1] \to [2] \to [3]$. For the third factor we have

$$\begin{aligned}\mathrm{pr}_3 \circ ((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= \mathrm{pr}_2 \circ (d_3 \times f_2)\\ &= f_2 = \mathrm{pr}_3 \circ (f_0 \times f_1 \times f_2).\end{aligned}$$

Hence $(f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) = f_0 \times f_1 \times f_2$ since these morphisms coincide after composing with each of the projections. This means that

For the middle left triangle we have

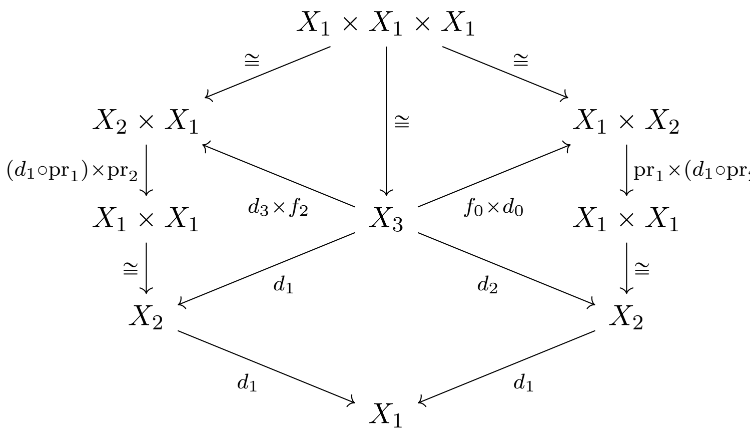
$$\begin{aligned}\mathrm{pr}_1 \circ ((d_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= (d_1 \circ \mathrm{pr}_1) \circ (d_3 \times f_2)\\ &= d_1 \circ d_3 = f_0 \circ d_1 = \mathrm{pr}_1 \circ (f_0 \times f_1) \circ d_1\end{aligned}$$

since $\delta^1 \circ \phi^0 = \delta^3 \circ \delta^1 \colon [1] \to [2] \to [3]$. We also have

$$\begin{aligned} \mathrm{pr}_2 \circ ((d_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= \mathrm{pr}_2 \circ (d_3 \times f_2) \\ &= f_2 = f_1 \circ d_1 = \mathrm{pr}_2 \circ (f_0 \times f_1) \circ d_1 \end{aligned}$$

since $\phi^2 = \delta^1 \circ \phi^1 \colon [1] \to [2] \to [3]$. Hence the morphisms $((d_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2)$ and $(f_0 \times f_1) \circ d_1$ coincide as morphisms into the product $X_1 \times X_1$.

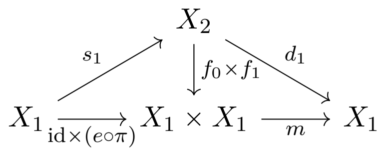
Inverting the isomorphisms $f_0 \times f_1 \times f_2$, $(f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2$, $\mathrm{pr}_1 \times (f_0 \circ \mathrm{pr}_2) \times (f_1 \circ \mathrm{pr}_2)$ and $f_0 \times f_1$, we obtain a commutative diagram



,

which contains the required associativity diagram.

In order to show that e is a right unit for m, we consider the diagram



.

Then the triangle on the right commutes by the definition of m.

On the left side we have

$$\mathrm{pr}_1 \circ (f_0 \times f_1) \circ s_1 = f_0 \circ s_1 = \mathrm{id}_{X_1}$$

since $\mathrm{id}_{[1]} = \sigma^1 \circ \phi^0 \colon [1] \to [2] \to [1]$. For the second factor we have

$$\mathrm{pr}_2 \circ (f_0 \times f_1) \circ s_1 = f_1 \circ s_1 = s_0 \circ d_1$$

since $\delta^1 \circ \sigma^0 = \sigma^1 \circ \phi^1 \colon [1] \to [1]$, and $s_0 \circ d_1 = e \circ \pi$ since π is the unique map into the terminal object $\ast$ and $e \colon \ast \xrightarrow{\cong} X_0 \xrightarrow{s_0} X_1$ by definition. Hence the left triangle is also commutative since $(f_0 \times f_1) \circ s_1$ and $\mathrm{id}_{X_1} \times (e \circ \pi)$ agree on both factors.

The commutativity of the above diagram yields $m \circ (\mathrm{id}_{X_1} \times (e \circ \pi)) = d_1 \circ s_1 = \mathrm{id}_{X_1}$, where the latter equality is a simplicial identity. Hence e is indeed a right unit for m.

Now one can analogously show that the diagram

$$\begin{array}{ccccc} & & X_2 & & \\ & \overset{s_2}{\nearrow} & \Big\downarrow{\scriptstyle f_0 \times f_1} & \overset{d_1}{\searrow} & \\ X_1 & \xrightarrow[(e\circ\pi)\times\mathrm{id}]{} & X_1 \times X_1 & \xrightarrow[m]{} & X_1 \end{array}$$

is also commutative. Hence e is also a left unit for m. □

Next, we want to see that this assignment is functorial.

Lemma A.5. *Let X, Y be special simplicial objects in $\mathscr{C}$, and $\gamma\colon X \to Y$ a morphism of simplicial objects, i. e. a natural transformation between the functors $X, Y\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$.*

Then we have

$$\prod_{i=0}^{n-1}(\gamma_1 \circ \mathrm{pr}_i) = \left(\prod_{i=0}^{n-1} f_i^Y\right) \circ \gamma_n \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} : X_1^n \to X_n \to Y_n \to Y_1^n$$

for all $n \in \mathbb{N}$.

Proof. For $n = 0$ the statement follows from the fact that there is a unique morphism between the terminal objects X_0 and Y_0.

For $n > 0$ we check the equality componentwise. Indeed, for $i_0 \in \{0, \dots, n-1\}$ we have

$$\begin{aligned} \mathrm{pr}_{i_0} \circ \left(\prod_{i=0}^{n-1} f_i^Y\right) \circ \gamma_n \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} &= f_{i_0}^Y \circ \gamma_n \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} \\ &= \gamma_1 \circ f_{i_0}^X \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} \\ &= \gamma_1 \circ \mathrm{pr}_{i_0} \\ &= \mathrm{pr}_{i_0} \circ \left(\prod_{i=0}^{n-1}(\gamma_1 \circ \mathrm{pr}_i)\right), \end{aligned}$$

where the second equality follows from the naturality of γ. □

Proposition A.6. *Let $\gamma\colon X \to Y$ be a morphism between special simplicial objects.*

Then $\gamma_1\colon X_1 \to Y_1$ is a morphism of monoid objects.

Proof. By the naturality of γ, the diagram

$$\begin{array}{ccc} * & \xrightarrow{\mathrm{id}} & * \\ {\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \cong} \\ X_0 & \xrightarrow{\gamma_0} & Y_0 \\ {\scriptstyle s_0^X}\Big\downarrow & & \Big\downarrow{\scriptstyle s_0^Y} \\ X_1 & \xrightarrow{\gamma_1} & Y_0 \end{array}$$

commutes, so γ_1 is compatible with the unit.

For the compatibility with the multiplication we consider the diagram

$$
\begin{array}{ccccccc}
X_1 \times X_1 & \xrightarrow{(f_0^X \times f_1^X)^{-1}} & X_2 & \xrightarrow{\gamma_2} & Y_2 & \xrightarrow{f_0^Y \times f_1^Y} & Y_2 \times Y_1 \\
 & \searrow^{m_X} & \downarrow{\scriptstyle d_1^X} & & \downarrow{\scriptstyle d_1^Y} & \swarrow^{m_Y} & \\
 & & X_1 & \xrightarrow{\gamma_1} & Y_1 & &
\end{array},
$$

which is commutative by the naturality of γ and the definition of m_X resp. m_Y. Now

$$
(f_0^Y \times f_1^Y) \circ \gamma_2 \circ (f_0^X \times f_1^X)^{-1} = (\gamma_1 \circ \mathrm{pr}_1) \times (\gamma_1 \circ \mathrm{pr}_2),
$$

by the previous lemma, which implies that the above diagram witnesses the compatibility of γ_1 with the multiplication. $\square$

Proposition A.4 and Proposition A.6 can be summarized as follows:

Corollary A.7. *The functor* $(\_)_1 \colon \mathbf{ss}\mathscr{C} \to \mathscr{C}$ *factors through the category* $\mathscr{C}$-**Mon** *of monoid objects in* $\mathscr{C}$.

In fact, $(\_)_1$ defines an equivalence of categories from $\mathbf{ss}\mathscr{C}$ to $\mathscr{C}$-**Mon**, where a quasi-inverse is given as follows:

For $M \in \mathrm{ob}(\mathscr{C}\text{-}\mathbf{Mon})$ with "multiplication" $m \colon M \times M \to M$ and "unit" $e \colon * \to M$, we define a (special) simplicial object X^M with $X_n^M = M^n$ for $n \in \mathbb{N}$, where the structure morphisms are given by

$$
d_i^{X^M} = \begin{cases} \prod_{j=2}^{n} \mathrm{pr}_j & i = 0 \\ \prod_{j=0}^{n-1} \mathrm{pr}_j & i = n \\ \left(\prod_{j=1}^{i-2} \mathrm{pr}_j\right) \times m \times \left(\prod_{j=i+1}^{n} \mathrm{pr}_j\right) & \text{otherwise} \end{cases}
$$

for $n \in \mathbb{N}_{>0}$ and $0 \leqslant i \leqslant n$ resp.

$$
s_i^{X^M} = \left(\prod_{j=1}^{i} \mathrm{pr}_j\right) \times (e \circ \pi) \times \left(\prod_{j=i+1}^{n} \mathrm{pr}_j\right)
$$

for $n \in \mathbb{N}$.

Given a morphism $f \colon M \to N$ of monoid objects in $\mathscr{C}$, we let $\gamma^f \colon X^M \Rightarrow X^N$ be given by

$$
\gamma_n^f \colon X_n^M = M^n \xrightarrow{\prod_{i=0}^{n} (f \circ \mathrm{pr}_i)} N^n = X_n^N
$$

for all $n \in \mathbb{N}$.

Since we don't use this statement we omit the tedious proof of the fact that the given is assignment is a well-defined functor which is indeed a quasi-inverse for $(\_)_1 \colon \mathbf{ss}\mathscr{C} \to \mathscr{C}$-**Mon**.

B Additive Categories

In this appendix we discuss certain descriptions of additive categories which lead to the additivity of stable derivators (see Corollary 3.1).

We begin with basic definitions and notations.

Definition B.1. A *preadditive category* is a category $\mathscr{A}$ s. t.

(i) $\mathscr{A}$ is pointed, i. e. has a zero object, which is an object 0 which is both initial and terminal,

(ii) binary (and hence all finite) products and coproducts exist in $\mathscr{A}$,

(iii) for any $X, Y \in \operatorname{ob}\mathscr{A}$, the morphism

$$(\mathrm{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \mathrm{id}_Y)\colon X \amalg Y \to X \times Y$$

is an isomorphism, where $0_{X,Y}\colon X \to 0 \to Y$ resp. $0_{Y,X}\colon Y \to 0 \to X$ is the unique morphism which factors through a zero object.

Notation B.2.

- Biproducts in the above sense will be denoted by $\_\oplus\_$.
- If X, Y, X' resp. Y' are objects of a preadditive category and $f_{X,X'}\colon X \to X'$, $f_{Y,X'}\colon Y \to X'$, $f_{X,Y'}\colon X \to Y'$ resp. $f_{Y,Y'}\colon Y \to Y'$ are some morphisms, then we denote the morphism

$$(f_{X,X'} \times f_{X,Y'}) \amalg (f_{Y,X'} \times f_{X,X'})\colon X \oplus Y \to X' \oplus Y'$$

by

$$\begin{pmatrix} f_{X,X'} & f_{Y,X'} \\ f_{X,Y'} & f_{Y,Y'} \end{pmatrix}.$$

Note that, using the universal properties of products and coproducts, any morphism $f\colon X \oplus Y \to X' \oplus Y'$ can be written as

$$f = \begin{pmatrix} \mathrm{pr}_{X'} \circ f \circ \mathrm{in}_X & \mathrm{pr}_{X'} \circ f \circ \mathrm{in}_Y \\ \mathrm{pr}_{Y'} \circ f \circ \mathrm{in}_X & \mathrm{pr}_{Y'} \circ f \circ \mathrm{in}_Y \end{pmatrix}.$$

Matrices of different sizes are constructed similarly.

- We will also use common abuses of notation such as denoting an identity morphism by 1 or a morphism that factors through a zero object by 0.

Next, we want to give an alternative description of preadditive categories.

Remark B.3. A preadditive category is enriched over the category **AbMon** of abelian monoids. Indeed, for any $X, Y \in \operatorname{ob}\mathscr{A}$, setting

$$f + g\colon X \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \oplus Y \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} Y$$

for $f, g \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$ yields an abelian monoid structure on $\mathrm{Hom}_{\mathscr{A}}(X,Y)$ with neutral element $0_{X,Y}$ and for any $X, Y, Z \in \mathrm{ob}\,\mathscr{A}$, the composition map

$$\_ \circ \_ \colon \mathrm{Hom}_{\mathscr{A}}(Y,Z) \times \mathrm{Hom}_{\mathscr{A}}(X,Y) \to \mathrm{Hom}_{\mathscr{A}}(X,Z)$$

is bilinear w. r. t. this "addition operation".

Furthermore, a straightforward computation shows that composing morphisms corresponds to multiplying their matrix representations.

Proposition B.4. *Let $\mathscr{A}$ be a category that has all finite products.*

Then $\mathscr{A}$ is preadditive if and only if it is enriched over the category **AbMon** *of abelian monoids, i. e. if all morphism sets of $\mathscr{A}$ have an abelian monoid structure s. t. composition is bilinear.*

Proof. A preadditive category has all finite products by definition and Remark B.3 means that it is also enriched over **AbMon**.

Now let $\mathscr{A}$ be a category that has all finite products and is enriched over **AbMon**. For $X, Y \in \mathrm{ob}\,\mathscr{A}$ let the "addition" in $\mathrm{Hom}_{\mathscr{A}}(X,Y)$ be denoted by $+_{X,Y}$ and its unit by $0_{X,Y}$.

$\mathscr{A}$ has in particular a terminal object $\ast$. The monoid structure on $\mathrm{Hom}_{\mathscr{A}}(\ast,\ast)$ is trivial since $\ast$ is a terminal object and any monoid with only one element is trivial. In particular, we have $\mathrm{id}_{\ast} = 0_{\ast,\ast}$.

For all $X \in \mathrm{ob}\,\mathscr{A}$, $\mathrm{Hom}_{\mathscr{A}}(\ast, X)$ has a monoid structure, hence is not empty. Now for any $f \colon \ast \to X$ we have $f = f \circ \mathrm{id}_{\ast} = f \circ 0_{\ast,\ast} = 0_{\ast,\ast}$ by the bilinearity of composition. Hence $\mathrm{Hom}_{\mathscr{A}}(\ast, X) \cong \{0_{\ast,X}\}$ for all $X \in \mathrm{ob}\,\mathscr{A}$, i.e. $\ast$ is also an initial object and therefore $\mathscr{A}$ is pointed. From now on 0 will denote a zero object in $\mathscr{A}$. Note that for any $X, Y \in \mathrm{ob}\,\mathscr{A}$, $0_{X,Y}$ is the unique morphism that factors through 0.

Let $X, Y \in \mathrm{ob}\,\mathscr{A}$. We want to endow $X \times Y$ with the structure of a coproduct of X and Y s. t. $(\mathrm{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \mathrm{id}_Y) = \mathrm{id}_{X \times Y}$. This enforces the structure morphisms of the coproduct to be $\mathrm{in}_1 := \mathrm{id}_X \times 0_{X,Y}$ and $\mathrm{in}_2 := 0_{Y,X} \times \mathrm{id}_Y$.

Given $Z \in \mathrm{ob}\,\mathscr{A}$ and morphisms $f_1 \colon X \to Z$ and $f_2 \colon Y \to Z$, define $f_1 \amalg f_2$ to be $(f_1 \circ \mathrm{pr}_1) +_{X \times Y, Z} (f_2 \circ \mathrm{pr}_2)$. Then we have indeed

$$\begin{aligned}
((f_1 \circ \mathrm{pr}_1) +_{X \times Y,Z} (f_2 \circ \mathrm{pr}_2)) \circ \mathrm{in}_1 &= \\
((f_1 \circ \mathrm{pr}_1) +_{X \times Y,Z} (f_2 \circ \mathrm{pr}_2)) \circ (\mathrm{id}_X \times 0_{X,Y}) &= \\
((f_1 \circ \mathrm{pr}_1) \circ (\mathrm{id}_X \times 0_{X,Y})) +_{X,Z} ((f_2 \circ \mathrm{pr}_2)) \circ (\mathrm{id}_X \times 0_{X,Y})) &= \\
(f_1 \circ \mathrm{id}_X) +_{X,Z} (f_0 \circ 0_{X,Y}) &= \\
f_1 +_{X,Z} 0_{X,Z} &= f_1
\end{aligned}$$

and similarly

$$\begin{aligned}
((f_1 \circ \mathrm{pr}_1) +_{X \times Y,Z} (f_2 \circ \mathrm{pr}_2)) \circ \mathrm{in}_2 &= \\
((f_1 \circ \mathrm{pr}_1) \circ (0_{Y,X} \times \mathrm{id}_Y)) +_{Y,Z} ((f_2 \circ \mathrm{pr}_2)) \circ (0_{Y,X} \times \mathrm{id}_Y)) &= \\
0_{Y,Z} +_{Y,Z} f_2 &= f_2.
\end{aligned}$$

Now let $f' \colon X \times Y \to Z$ be another morphism s. t. $f' \circ \mathrm{in}_1 = f_1$ and $f' \circ \mathrm{in}_2 = f_2$.

We first claim that

$$\mathrm{id}_{X\times Y} = (\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2).$$

Indeed, by the bilinearity of composition we have

$$\begin{aligned} \mathrm{pr}_1 \circ ((\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2)) = \\ (\mathrm{pr}_1 \circ (\mathrm{pr}_1 \times 0_{X\times Y,Y})) +_{X\times Y,X} (\mathrm{pr}_1 \circ (0_{X\times Y,X} \times \mathrm{pr}_2)) = \\ \mathrm{pr}_1 +_{X\times Y,X} 0_{X\times Y,X} = \mathrm{pr}_1 \end{aligned}$$

and similarly

$$\mathrm{pr}_2 \circ ((\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2)) = \mathrm{pr}_2.$$

Hence the two morphisms coincide since they agree on both factors.

Moreover, note that we have

$$\mathrm{pr}_1 \circ (\mathrm{id}_X \times 0_{X,Y}) \circ \mathrm{pr}_1 = \mathrm{pr}_1$$

and

$$\mathrm{pr}_2 \circ (\mathrm{id}_X \times 0_{X,Y}) \circ \mathrm{pr}_1 = 0_{X,Y} \circ \mathrm{pr}_1 = 0_{X\times Y,Y},$$

which means that $\mathrm{in}_1 \circ \mathrm{pr}_1 = (\mathrm{id}_X \times 0_{X,Y}) \circ \mathrm{pr}_1 = \mathrm{pr}_1 \times 0_{X\times Y,Y}$ since these morphisms agree on both factors. Similarly, we also have $\mathrm{in}_2 \circ \mathrm{pr}_2 = (0_{Y,X} \times \mathrm{id}_Y) \circ \mathrm{pr}_2 = 0_{X\times Y,X} \times \mathrm{pr}_2$.

Again using the bilinearity of composition, these yield

$$\begin{aligned} f' \circ \mathrm{id}_{X\times Y} &= f' \circ ((\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2)) \\ &= (f' \circ (\mathrm{pr}_1 \times 0_{X\times Y,Y})) +_{X\times Y,X\times Y} (f' \circ (0_{X\times Y,X} \times \mathrm{pr}_2)) \\ &= (f' \circ \mathrm{in}_1 \circ \mathrm{pr}_1) +_{X\times Y,X\times Y} (f' \circ \mathrm{in}_2 \circ \mathrm{pr}_2) \\ &= (f_1 \circ \mathrm{pr}_1) +_{X\times Y,X\times Y} (f_2 \circ \mathrm{pr}_2) = f_1 \amalg f_2. \end{aligned}$$

Hence in_1 and in_2 do endow $X \times Y$ with a suitable coproduct structure. □

Using this characterization we obtain a generic class of preadditive categories.

Proposition B.5. *Let $\mathscr{C}$ be a category which has all finite products.*

*Then the category $\mathscr{C}$-**AbMon** of abelian monoid objects (with homomorphisms of monoid objects between them) is a preadditive category.*

Proof. First we note that $\mathscr{C}$-**AbMon** can alternatively be described as follows:

$$\mathrm{ob}(\mathscr{C}\text{-}\mathbf{AbMon}) = \{M \in \mathrm{ob}\,\mathscr{C} \mid \mathrm{Hom}_{\mathscr{C}}(\_, M) \text{ factors through } \mathbf{AbMon}\},$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}\text{-}\mathbf{AbMon}}(M,N) = \{f \in \mathrm{Hom}_{\mathscr{C}}(M,N) \mid\ & \\ & (f_*)_X \colon \mathrm{Hom}_{\mathscr{C}}(X,M) \to \mathrm{Hom}_{\mathscr{C}}(X,N) \\ & \text{is a morphisms of (abelian) monoids for all} \\ & X \in \mathrm{ob}\,\mathscr{C}\}\end{aligned}$$

for any $M, N \in \mathrm{ob}(\mathscr{C}\text{-}\mathbf{AbMon})$.

Hence, for any $M, N \in \mathrm{ob}(\mathscr{C}\text{-}\mathbf{AbMon})$, $\mathrm{Hom}_{\mathscr{C}\text{-}\mathbf{AbMon}}(M,N)$ has an abelian monoid structure given by pointwise addition and units on the level of represented functors. Then compisition is bilinear w. r. t. this addition since morphisms between monoid objects are chosen to preserve the addition on the homomorphism sets.

This means that $\mathscr{C}$-**AbMon** enriched over **AbMon** and hence a preadditive category by the previous proposition. ☐

Remark B.6. Remark B.3 implies also that any object X of a preadditive category has the structure of an abelian monoid object given by the codiagonal morphism $\nabla := \begin{pmatrix} 1 & 1 \end{pmatrix} : X \oplus X \to X$ and the "unit" $0 \to X$. Dually, X has also the structure of a coabelian comonoid object given by the diagonal morphism $\Delta := \begin{pmatrix} 1 \\ 1 \end{pmatrix} : X \oplus X \to X$ and the "counit" $X \to 0$.

In fact, for a preadditive category $\mathscr{A}$, the functor $\mathscr{A} \to \mathscr{A}$-**AbMon** which endows an object with the above abelian monoid structure is an equivalence of categories. However, we will neither use nor prove this statement.

As the prefix "pre" suggests, preadditive categories are not quite what we are looking for. We get to the concept of additive categories by requiring that additive inverses of morphisms exist.

Proposition B.7. *Let $\mathscr{A}$ be a preadditive category.*

Then the following are equivalent:

(i) For any $X \in \mathrm{ob}\,\mathscr{A}$, the "shear morphism"

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : X \oplus X \to X \oplus X$$

is an isomorphism.

(ii) For any $X \in \mathrm{ob}\,\mathscr{A}$, the identity morphism id_X has an additive inverse in $\mathrm{End}_{\mathscr{A}}(A)$.

(iii) For any $X, Y \in \mathrm{ob}\,\mathscr{A}$, each $f \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$ has an additive inverse.

(iv) For any $X \in \mathrm{ob}\,\mathscr{A}$, the abelian monoid object $(X, \nabla, 0 \to X)$ is an (abelian) group object.

(v) *For any $X \in \operatorname{ob}\mathcal{A}$, the coabelian comonoid object $(X, \Delta, X \to 0)$ is a (coabelian) cogroup object.*

Proof. "$(i) \Rightarrow (ii)$": Let the inverse of the shear morphism of X be given by

$$\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} : X \oplus X \to X \oplus X.$$

Then we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} = \begin{pmatrix} j_{1,1} + j_{2,1} & j_{1,2} + j_{2,2} \\ j_{2,1} & j_{2,2}. \end{pmatrix}$$

Hence $j_{1,2} = 0_{X,X}$ and $j_{1,1} = j_{2,2} = \operatorname{id}_X$. This yields

$$\operatorname{id}_X + j_{1,2} = j_{2,2} + j_{1,2} = 0_{X,X},$$

so $j_{1,2}$ is an additive inverse of id_X.

"$(ii) \Rightarrow (iii)$": Let $-\operatorname{id}_X$ be an additive inverse for id_X. Then the bilinearity of composition yields

$$f + f \circ (-\operatorname{id}_X) = f \circ \operatorname{id}_X + f \circ (-\operatorname{id}_X) = f \circ (\operatorname{id}_X + (-\operatorname{id}_X)) = f \circ 0_{X,X} = 0_{X,Y},$$

i. e. $f \circ (-\operatorname{id}_X)$ is an additive inverse for f.

"$(iii) \Rightarrow (iv)$": Note that X is a group object in $\mathcal{A}$ if and only if its represented functor $\operatorname{Hom}_{\mathcal{A}}(\_, X)$ factors through the category **Grp** of groups. Since X is an abelian monoid object, we already know that $\operatorname{Hom}_{\mathcal{A}}(\_, X)$ factors through **AbMon**. Now the fact that for each $Y \in \operatorname{ob}\mathcal{A}$ each $f \in \operatorname{Hom}_{\mathcal{A}}(Y, X)$ has an additive inverse implies that the abelian monoids $(\operatorname{Hom}_{\mathcal{A}}(Y, X), +_{Y,X}, 0_{Y,X})$ are in fact abelian groups. Since all monoid homomorphisms between groups are already homomorphisms of groups, this means that $\operatorname{Hom}_{\mathcal{A}}(\_, X)$ factors through the category of (abelian) groups.

"$(iv) \Rightarrow (v)$": If X is a group object with the "multiplication" given by ∇, there exists a morphism $j\colon X \to X$ s. t.

$$0_{X,X} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} = \operatorname{id}_X \circ \operatorname{id}_X + \operatorname{id}_X \circ j = \operatorname{id}_X + j.$$

Hence for comonoid structure on X we obtain

$$\begin{pmatrix} 1 & j \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \operatorname{id}_X \circ \operatorname{id}_X + j \circ \operatorname{id}_X = \operatorname{id}_X + j = 0_{X,X}.$$

A similar argument shows that the fact that j is also a "left inverse for the multiplication of X" implies that j is also a "left inverse for the comultiplication of X". In total, we obtain that $(X, \Delta, X \to 0, j)$ is a cogroup object.

"$(v) \Rightarrow (i)$": Let $j\colon X \to X$ be the "coinverse" morphism w. r. t. Δ. Then calculations similar to the ones in the proof of the previous implication yield that $\mathrm{id}_X + j = 0_{X,X} = j + \mathrm{id}_X$. Hence we obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & j+1 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+j \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathrm{id}_{X \oplus X} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we see that the shear morphism is an isomorphism. □

Definition B.8. A preadditive category is called *additive* if it satisfies one (and hence all) of the conditions in the previous proposition.

Additive categories also have an alternative characterization similar to the one in Proposition B.4 for preadditive categories.

Corollary B.9. *A category $\mathscr{A}$ which has all finite products is additive if and only if it is enriched over the category* **Ab** *of abelian groups.*

Proof. If $\mathscr{A}$ is additive then the condition (*iii*) in Proposition B.7 means that $\mathscr{A}$ is enriched not only over **AbMon**, but even over **Ab** since additive inverses exist in homomorphism monoids and (bi)linear maps of abelian groups are exactly (bi)linear maps of underlying monoids.

Conversely, if $\mathscr{A}$ is enriched over **Ab**, then $\mathscr{A}$ is preadditive by Proposition B.4 and the condition (*iii*) in Proposition B.7 is fulfilled since additive inverses in all homomorphism monoids exist. □

This characterization yields a generic class of examples which is used in the proof of Corollary 3.1.

Corollary B.10. *Let $\mathscr{C}$ be a category which has all finite products.*

*Then the category $\mathscr{C}$-***Ab** *of abelian group objects (with homomorphisms of group objects between them) is an additive category.*

Proof. Just as the previous corollary, this follows immediately from Proposition B.4 and the condition (*iii*) in Proposition B.7. □

Lastly, let us note that as in the case of preadditive categories, an additive category $\mathscr{A}$ is in fact equivalent to the category of abelian group objects in $\mathscr{A}$.

References

[1] Groth, M. Derivators, pointed derivators and stable derivators. *Algebr. Geom. Topol. 13*, 1 (2013), 313–374.

[2] Groth, M. Selected topics in topology: Derivators. `http://guests.mpim-bonn.mpg.de/mgroth/teaching/derivators14/Groth_derivators-NOTES.pdf`, 2015. Accessed version published on 2015-01-27.

Index